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Arithmetic partial differential equations, II

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Abstract

We continue the study of arithmetic partial differential equations initiated in [7] by classifying "arithmetic convection equations" on modular curves, and by describing their space of solutions. Certain of these solutions involve the Fourier expansions of the Eisenstein modular forms of weight 4 and 6, while others involve the Serre–Tate expansions (Mori, 1995 [13], Buium, 2003 [4]) of the same modular forms; in this sense, our arithmetic convection equations can be seen as "unifying" the two types of expansions. The theory can be generalized to one of "arithmetic heat equations" on modular curves, but we prove that they do not carry "arithmetic wave equations." Finally, we prove an instability result for families of arithmetic heat equations converging to an arithmetic convection equation.

Keywords: Differential algebra; Fermat quotients; Modular forms

1. Introduction

1.1. Main concepts and results

This article is a direct continuation of [7], and for the purpose of the current introduction, we shall assume some familiarity with the introduction to our earlier work. In there we developed an arithmetic analogue of the theory of partial differential equations in space–time, where the rôle of the spatial derivative is played by a Fermat quotient operator δ_p with respect to a fixed prime p, while the rôle of derivative with respect to (the "exponential" q of) time is played by the usual derivation operator δ_q. (We view the operator δ_p as a derivative with respect to p; the

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ordinary differential theory based on this interpretation was initiated in [2].) The rings on which δ_p and δ_q act, which satisfy certain compatibility conditions, are called $\{\delta_p, \delta_q\}$-rings. If X is a scheme of finite type over a p-adically complete $\{\delta_p, \delta_q\}$-ring A, we define a partial differential operator as a map $f : X(A) \to A$ given locally, in the Zariski topology, by a restricted p-adic power series in the affine coordinates x and finitely many of its iterated "derivatives" $\delta_p^i \delta_q^j x$.

Given a partial differential operator $f : X(A) \to A$, we may consider the *equation* $f = 0$, and its *space of solutions* $\mathcal{U} := f^{-1}(0) \subset X(A)$. It is natural to restrict our attention to special cases where an appropriate notion of *linearity* for f can be defined, and then to investigate such linear operators. In our earlier work [7] we concentrated primarily on the case when X is a commutative group scheme of dimension 1 (especially $\mathbb{G}_a$, $\mathbb{G}_m$, or $X = E$, an elliptic curve) and f is a homomorphism (to the additive group of A); the homomorphism condition was meant to be an analogue of linearity for classical partial differential operators in analysis. Three classes of these equations ended up playing a central rôle in our study. They can be viewed as arithmetic analogues of convection, heat, and wave equations, respectively.

In the current article we consider a modular analogue of the situation above, where we now take X to be the total space $\mathbb{M}_{\Gamma_1(N)}$ of a natural $\mathbb{G}_m$-torsor over a modular curve, or a "stacky" version of such an object. (With precision, $\mathbb{M}_{\Gamma_1(N)}$ classifies elliptic curves, possibly with some level structure, together with an invertible 1-form. In our work here, we present the theory without reference to modular curves, in a purely stacky manner, in the style of [11] and [3]; the translation in terms of modular curves is straightforward.) We shall assume that the function f is homogeneous of a certain weight with respect to the action of the multiplicative group on X; this will make of f a *partial differential modular form* (cf. [3] for the ordinary differential case). For such an f, the space of solutions $\mathcal{U}$ has a $\mathbb{G}_m$-action. In addition, we will assume that f has a certain covariance with respect to the action of Hecke correspondences; these fs will be called *isogeny covariants* (again, cf. [3] for the ordinary differential case), and for them, the space of solutions $\mathcal{U}$ is also saturated with respect to "isogeny." Isogeny covariant forms of *covariance degree* 1 will be viewed as arithmetic analogues, in our context here, of linear partial differential equations in analysis. For forms of even integral weight m, the covariance degree equals $-m/2$, and so linearity corresponds to weight -2. In what follows, we will consider a generalization of this situation, where the fs are allowed to have singularities along the supersingular locus "$E_{p-1} = 0$" of X.

In parallel with our results in [7], we address two main problems here: the first is to find all possible linear partial differential equations, and the second is to describe the space of solution $\mathcal{U}$ of any one such. We deal with these two problems in Sections 2 and 3, respectively. The ordinary differential theory [3] in the arithmetic p-direction (respectively, in the geometric q-direction) provides order 1 isogeny covariant forms f_p^1 and f_p^∂ involving δ_p only (respectively, an order 1 isogeny covariant form f_q^1 involving δ_q only). Both f_q^1 and the product $f_{p,-2}^1 := f_p^\partial f_p^1$ have weight -2. In Section 2 of we prove that f_q^1 and $f_{p,-2}^1$ form a basis of the space of isogeny covariant forms with singularities along $E_{p-1} = 0$, of weight -2, and order 1. These forms should be viewed as "arithmetic convection equations." In Section 3 we analyze the spaces of solutions $\mathcal{U}$ for such forms. We are particularly interested in two subsets of $\mathcal{U}$, denoted by $\mathcal{U}_{bad}$ and $\mathcal{U}_{good}$, respectively, that consist of solutions with bad and good reductions at infinity, and which satisfy certain non-degeneracy conditions. If f is a linear combination of the form $f = f_q^1 + \lambda f_{p,-2}^1$, then, for λ invertible, we have the following "quantization" phenomenon (reminiscent of our theory in [7]): if the parameter λ is not a positive integer times a certain fixed quantity, then the set $\mathcal{U}_{bad}$ is a union of $\mathbb{G}_m$-orbits of one basic solution (that can be expressed in terms of the Fourier expansions of Eisenstein modular forms of weights 4 and 6) and of their shifts by roots of unity.

For the exceptional values of λ, the space $\mathcal{U}_{bad}$ has a different, more complicated structure, which we shall completely describe in terms of closed form formulae. With the set $\mathcal{U}_{good}$ we encounter a similar phenomenon: for λ not a positive integer times a certain fixed quantity, $\mathcal{U}_{good}$ consists of $\mathbb{G}_m$-orbits of "stationary" solutions. For the exceptional values of λ, the set $\mathcal{U}_{good}$ has again a richer but completely understood structure. The rôle of the Fourier series of Eisenstein forms of weights 4 and 6 is played now, in the good reduction case, by the "Serre–Tate expansions" [13,4] of these modular forms. In this way, both the Fourier expansions and the Serre–Tate expansions appear in the solutions of the same arithmetic partial differential equation; in a sense, these two types of expansions are "unified" by the said equation. Furthermore, we will discover "canonical" 1-to-1 correspondences between certain sets of bad reduction solutions and certain sets of good reduction solutions of appropriate pairs of arithmetic convection equations. A similar result holds for "arithmetic heat equations" that are "close to" arithmetic convection equations.

Although arithmetic convection and heat equations exist, we will prove also that, in a suitable sense, modular curves do not carry "arithmetic wave equations."

We shall end with the discovery of an "instability phenomenon." As "arithmetic heat equations" converge to a given "arithmetic convection equation," the solutions of the arithmetic heat equations with given boundary conditions *do not converge* to the corresponding solutions of the arithmetic convection equation. This situation does not seem to have a parallel when viewing solutions of heat versus convection equations in real analysis.

1.2. Review of terminology and notation

We end this introduction by recalling some of the basic concepts and notation from [7] that we will need later on, as well as introducing the ring of weights.

A derivation from a ring A to an A-algebra B is a map $\delta : A \to B$ such that $\delta(x+y) = \delta x + \delta y$ and $\delta(xy) = x\delta y + y\delta x$. If p is a prime, a p-derivation is a map $\delta = \delta_p : A \to B$ such that

$$\delta(x+y) = \delta x + \delta y + C_p(x,y),$$
$$\delta(xy) = x^p\delta y + y^p\delta x + p\delta x\delta y,$$

where $C_p(X,Y)$ stands for the polynomial with $\mathbb{Z}$-coefficients

$$C_p(X,Y) := \frac{X^p + Y^p - (X+Y)^p}{p}.$$

If δ_p is a p-derivation then $\phi_p : A \to B$ defined by

$$\phi_p x := x^p + p\delta_p x,$$

is always a ring homomorphism. A $\{\delta_p, \delta_q\}$-ring is a ring A equipped with a derivation $\delta_q : A \to A$ and a p-derivation $\delta_p : A \to A$, such that

$$\delta_q\delta_p x = p\delta_p\delta_q x + (\delta_q x)^p - x^{p-1}\delta_q x. \tag{1}$$

A morphism of $\{\delta_p, \delta_q\}$-rings is a ring homomorphism that commutes with δ_p and δ_q. A $\{\delta_p, \delta_q\}$-*prolongation sequence* is a sequence of rings $S^* = \{S^n\}_{n\geqslant 0}$ equipped with ring homomorphisms $\varphi : S^n \to S^{n+1}$ (used to view each S^{n+1} as an S^n-algebra), p-derivations $\delta_p : S^n \to S^{n+1}$, and derivations $\delta_q : S^n \to S^{n+1}$ satisfying condition (1), and such that $\delta_p \circ \varphi = \varphi \circ \delta_p$, and $\delta_q \circ \varphi = \varphi \circ \delta_q$, respectively. A *morphism* $u^* : S^* \to \tilde{S}^*$ of $\{\delta_p, \delta_q\}$-prolongation sequences is a sequence $u^n : S^n \to \tilde{S}^n$ of ring homomorphisms such that $\delta_p \circ u^n = u^{n+1} \circ \delta_p$, $\delta_q \circ u^n = u^{n+1} \circ \delta_q$, and $\varphi \circ u^n = u^{n+1} \circ \varphi$. If A is a $\{\delta_p, \delta_q\}$-ring, then we can attach to it a prolongation sequence (which we still denote by A) by taking $A^n = A$, $\varphi = id$. A prolongation sequence S^* over a $\{\delta_p, \delta_q\}$-ring A is a morphism of $\{\delta_p, \delta_q\}$-prolongation sequences $A \to S^*$.

Throughout our work we fix a prime $p \geqslant 5$. A basic example of $\{\delta_p, \delta_q\}$-ring is $A = R := R_p = (\mathbb{Z}_p^{ur})\widehat{\ }$, with $\delta_q = 0$, and

$$\delta_p x = \frac{\phi_p(x) - x^p}{p},$$

where ϕ_p is the unique lift of the Frobenius map to R/pR. Here, and in the sequel, the superscript $\widehat{\ }$ denotes p-adic completion. Another basic example of $\{\delta_p, \delta_q\}$-ring that plays a rôle here is $A = R((q))\widehat{\ }$, where q is an indeterminate over R, $R((q)) := R[\![q]\!][q^{-1}]$, $\delta_q F = q\frac{\partial F}{\partial q}$, and

$$\delta_p F = \frac{F^{(\phi_p)}(q^p) - (F(q))^q}{p},$$

where the superscript (ϕ_p) means twisting coefficients by $\phi_p : R \to R$. A basic example of prolongation sequence is as follows. For a $\{\delta_p, \delta_q\}$-ring A and z a variable, we define

$$A\big[z^{(\leqslant r)}\big]\widehat{\ } := A\big[z^{(i,j)}\big|_{i+j\leqslant r}\big]\widehat{\ }$$

where $i, j \geqslant 0$, $z^{(0,0)} = z$. Then there is a unique structure of a $\{\delta_p, \delta_q\}$-prolongation sequence on this sequence, extending that of A, such that $\delta_p z^{(i,j)} = z^{(i+1,j)}$ and $\delta_q z^{(i,j)} = z^{(i,j+1)}$. A similar definition can be given for z a tuple of variables. More generally, if $g \in A[z]\widehat{\ } \setminus (p)$, we may consider the sequence $A[z^{(\leqslant r)}, g^{-1}]\widehat{\ }$; this has a unique structure of $\{\delta_p, \delta_q\}$-prolongation sequence extending the one just defined.

We will also need the ring of *weights* $W := \mathbb{Z}[\phi]$, the polynomial ring in the "variable" ϕ; we let W_+ denote the semigroup of all $w = \sum_{i=0}^r a_i\phi^i \in W$ with $a_i \geqslant 0$. We define $\mathrm{ord}(w) = r$ if $a_r \neq 0$, $\deg(w) = \sum a_i$. If λ is an invertible element in a $\{\delta_p, \delta_q\}$-ring A and $w \in W$ (or if λ is not necessarily invertible but $w \in W_+$), we set $\lambda^w = \prod \phi_p^i(\lambda)^{a_i}$. The same notation can be introduced for $\{\delta_p, \delta_q\}$-prolongation sequences.

If instead of $\{\delta_p, \delta_q\}$ we use only δ_p (respectively δ_q) we are led to obvious notions of δ_p-ring, δ_p-prolongation sequence, etc. (respectively δ_q-ring, δ_q-prolongation sequence, etc.).

2. Equations

In this section we introduce and classify some of our arithmetic analogues of linear partial differential equations on modular curves.

2.1. $\{\delta_p, \delta_q\}$-modular forms

The following definitions are analogues in the partial differential case of some definitions given in [3]; they follow the viewpoint put forward in [11] that treated the "non-differential" setting.

Let N be a positive integer that is not divisible by p, and that we shall fix throughout the paper. Given any ring S where 6 is an invertible element, we denote by $\mathbb{M}(\Gamma_1(N), S)$ the set of all triples $(E/S, \alpha, \omega)$ where E/S is an elliptic curve, ω is an invertible 1-form on E, and $\alpha : (\mathbb{Z}/N\mathbb{Z})_S \to E$ is a closed immersion of group schemes (referred to as a $\Gamma_1(N)$-*level structure*). When $N = 1$, we shall usually drop α and $\Gamma_1(1)$ from the notation. In particular, we write $\mathbb{M}(S)$ for $\mathbb{M}(\Gamma_1(1), S)$.

Notice that we have an identification

$$\mathbb{M}(S) = \left\{(a, b) \in S \times S \;\middle|\; 4a^3 + 27b^2 \in S^\times\right\}$$

given by attaching to each pair $(a, b) \in \mathbb{M}(S)$, the pair (E, ω) where E is the projective closure of the affine curve $y^2 = x^3 + ax + b$, and $\omega = dx/y$.

There is an action of $\mathbb{G}_m(S) = S^\times$ on $\mathbb{M}(\Gamma_1(N), S)$ via multiplication on the ω component. The induced action on $\mathbb{M}(S)$ is given by the formula $\lambda(a, b) = (\lambda^{-4}a, \lambda^{-6}b)$.

Remark 2.1. Notice that when $N > 4$, $\mathbb{M}(\Gamma_1(N), S)$ is the set of S-points of a natural $\mathbb{G}_m$-torsor $\mathbb{M}_{\Gamma_1(N)}$ over the modular curve $Y_1(N)$. The same is true when $N \leqslant 4$, but only in a "stacky sense." Cf. [11,8]. For $N = 1$, $\mathbb{M}(S)$ is, of course, the set of S-points of the *modular scheme*

$$\mathbb{M} = \mathbb{M}_\Gamma := \operatorname{Spec} \mathbb{Z}[1/6]\left[a_4, a_6, \left(4a_4^3 + 27a_6^2\right)^{-1}\right],$$

where a_4, a_6 are indeterminates and Γ stands for $\mathrm{SL}_2(\mathbb{Z}) = \Gamma_1(1)$.

Let us fix a weight w in the ring of weights W, with $\operatorname{ord}(w) \leqslant r$, and a $\{\delta_p, \delta_q\}$-ring A that is Noetherian, p-adically complete, and an integral domain of characteristic zero.

Definition 2.2. A $\{\delta_p, \delta_q\}$-*modular form* over A, of weight $w \in W$ and order r on $\Gamma_1(N)$, is a rule f that associates to any $\{\delta_p, \delta_q\}$-prolongation sequence S^* of Noetherian, p-adically complete rings over A, and any triple $(E/S^0, \alpha, \omega) \in \mathbb{M}(\Gamma_1(N), S^0)$, an element $f(E/S^0, \alpha, \omega, S^*) \in S^r$ subject to the following conditions:

(1) $f(E/S^0, \alpha, \omega, S^*)$ depends on the isomorphism class of $(E/S^0, \alpha, \omega)$ only.
(2) The formation of $f(E/S^0, \alpha, \omega, S^*)$ commutes with base change $u^* : S^* \to \tilde{S}^*$, that is to say,

$$f\left(E \otimes_{S^0} \tilde{S}^0/\tilde{S}^0, \alpha \otimes \tilde{S}^0, u^{0*}\omega, \tilde{S}^*\right) = u^r\left(f\left(E/S^0, \alpha, \omega, S^*\right)\right).$$

(3) $f(E/S^0, \alpha, \lambda\omega, S^*) = \lambda^{-w} \cdot f(E/S^0, \alpha, \omega, S^*)$ for all $\lambda \in (S^0)^\times$.

We denote by $M^r_{pq}(\Gamma_1(N), A, w)$ the A-module of all $\{\delta_p, \delta_q\}$-modular forms over A of weight $w \in W$ and order r on $\Gamma_1(N)$.

We have natural maps

$$\varphi : M^r_{pq}\left(\Gamma_1(N), A, w\right) \to M^{r+1}_{pq}\left(\Gamma_1(N), A, w\right)$$

that send any f into $\varphi \circ f$, and we have natural maps

$$\phi : M^r_{pq}\big(\Gamma_1(N), A, w\big) \to M^{r+1}_{pq}\big(\Gamma_1(N), A, \phi w\big)$$

that send any f into $f^\phi := \phi \circ f$. Finally, notice that for any integer N' such $N'|N$, there are natural maps

$$M^r_{pq}\big(\Gamma_1\big(N'\big), A, w\big) \to M^r_{pq}\big(\Gamma_1(N), A, w\big).$$

In particular, for $N' = 1$, we have a map

$$M^r_{pq}(A, w) \to M^r_{pq}\big(\Gamma_1(N), A, w\big).$$

Following [3] verbatim, we obtain the following description of $M^r_{pq}(A, w)$. Let a_4, a_6, and Λ be variables, and set $\Delta := -2^6 a_4^3 - 2^4 3^3 a_6^2$. Then $M^r_{pq}(A, w)$ identifies with the set of all power series f in

$$M^r_{pq} := A\big[a_4^{(\leqslant r)}, a_6^{(\leqslant r)}, \Delta^{-1}\big]\widehat{\ }$$

such that

$$\begin{aligned} &f\big(\dots, \delta_p^i \delta_q^j\big(\Lambda^4 a_4\big), \dots, \delta_p^i \delta_q^j\big(\Lambda^6 a_6\big), \dots, \Lambda^{-12}\Delta^{-1}\big) \\ &\quad = \Lambda^w f\big(\dots, \delta_p^i \delta_q^j a_4, \dots, \delta_p^i \delta_q^j a_6, \dots, \Delta^{-1}\big). \end{aligned}$$

Given $f \in M^r_{pq}(A, w)$, we shall use the same symbol and still denote by $f \in M^r_{pq}$ the corresponding series. The map

$$\mathbb{M}(A) \to A \tag{2}$$

defined by

$$(a, b) \mapsto f\big(\dots, \delta_p^i \delta_q^j a, \dots, \delta_p^i \delta_q^j b, \dots, \Delta^{-1}\big),$$

will be denoted by f; it has the property that

$$f\big(\lambda^4 a, \lambda^6 b\big) = \lambda^w f(a, b)$$

for all $\lambda \in A^\times$.

A form $f \in M^r_{pq}(\Gamma_1(N), A, w)$ will be called *essentially of level one* if the rule f does not depend on the variable α. We denote by $M^r_{pq}(\Gamma_1(N), A, w)_1$ the subspace of $M^r_{pq}(\Gamma_1(N), A, w)$ consisting of forms that are essentially of level one.

Lemma 2.3. *The natural maps $M^r_{pq}(A, w) \to M^r_{pq}(\Gamma_1(N), A, w)_1$ are isomorphisms.*

Proof. We prove surjectivity. Injectivity follows similarly.

Let

$$\tilde{f} \in M^r_{pq}\big(\Gamma_1(N), A, w\big)_1,$$

and consider a triple (E, ω, S^*). There is a Galois étale S^0-algebra $\tilde{S}^0$ such that $E \otimes_{S^0} \tilde{S}^0$ has a $\Gamma_1(N)$-level structure α. As in (3.15) of [3], $S^* \otimes_{S^0} \tilde{S}^0$ has a unique structure of $\{\delta_p, \delta_q\}$-prolongation sequence extending that of S^*. Set

$$f\big(E, \omega, S^*\big) := \tilde{f}\big(E \otimes_{S^0} \tilde{S}^0, \omega \otimes 1, \alpha, S^* \otimes_{S^0} \tilde{S}^0\big).$$

This quantity does not depend on the choice of α, and by functoriality is seen to belong to S^r (rather than $S^r \otimes_{S^0} \tilde{S}^0$). Clearly f is mapped into $\tilde{f}$. $\square$

Now for any $f \in M^r_{pq}(A, w)$, we may consider the map $f : \mathbb{M}(A) \to A$ in (2). We say that $f = 0$ is an equation of weight w, and $f^{-1}(0) \subset \mathbb{M}(A)$ is its *space of solutions*. Notice that $f^{-1}(0)$ is invariant under the $\mathbb{G}_m$-action: if $(a, b) \in f^{-1}(0)$ and $v \in A^\times$, then $(v^4 a, v^6 b) \in f^{-1}(0)$. In what follows, we will impose another property on f, *isogeny covariance*, which gives the space of solutions an extra symmetry property with respect to "Hecke correspondences."

2.2. *Isogeny covariance*

Let $f \in M^r_{pq}(\Gamma_1(N), A, w)$ be a $\{\delta_p, \delta_q\}$-modular form of weight $w = \sum n_i \phi^i$ on $\Gamma_1(N)$. Assume $\deg(w) := \sum n_i$ is even. Generalizing the level one definition in [3], we say that f is *isogeny covariant* if for any $\{\delta_p, \delta_q\}$-prolongation sequence S^*, any triples $(E_1, \alpha_1, \omega_1)$, $(E_2, \alpha_2, \omega_2) \in \mathbb{M}(\Gamma_1(N), S^0)$, and any isogeny $u : E_1 \to E_2$ of degree prime to p, with $\omega_1 = u^* \omega_2$ and $u \circ \alpha_1 = \alpha_2$, we have that

$$f\big(E_1, \alpha_1, \omega_1, S^*\big) = \deg(u)^{-\deg(w)/2} \cdot f\big(E_2, \alpha_2, \omega_2, S^*\big).$$

The number $-\deg(w)/2$ will be called the *covariance degree* of f. Forms of covariance degree 1 will be called *linear*; they should be viewed as the analogues, in our current context, of the linear differential operators in analysis.

We denote by $I^r_{pq}(\Gamma_1(N), A, w)$ the space of isogeny covariant forms belonging to the space $M^r_{pq}(\Gamma_1(N), A, w)$. We denote by $I^r_{pq}(\Gamma_1(N), A, w)_1$ the subspace of $I^r_{pq}(\Gamma_1(N), A, w)$ consisting of forms which are essentially of level one. By Lemma 2.3, it follows that the natural maps $I^r_{pq}(A, w) \to I^r_{pq}(\Gamma_1(N), A, w)_1$ are isomorphisms.

Notice that if f is isogeny covariant, then its space of solutions $f^{-1}(0)$ has the following extra symmetry (which is morally a symmetry with respect to Hecke correspondences). For if we assume $y^2 = x^3 + a_1 x + b_1$ and $y^2 = x^3 + a_2 x + b_2$ are two elliptic curves with coefficients in A, and that there exists an isogeny over A of degree prime to p between them that pulls back dx/y into dx/y, then $(a_1, b_1) \in f^{-1}(0)$ if, and only if, $(a_2, b_2) \in f^{-1}(0)$.

2.3. Variants

If in the definitions above we use δ_p-rings and δ_p-prolongation sequences instead of $\{\delta_p, \delta_q\}$-rings and $\{\delta_p, \delta_q\}$-prolongation sequences, we obtain the concept of *δ_p-modular form* (over a δ_p-ring A); in that case, we denote the corresponding spaces by $M_p^r(\Gamma_1(N), A, w)$ and $I_p^r(\Gamma_1(N), A, w)$, respectively. For $N = 1$, these δ_p-modular forms are the δ-modular forms in the arithmetic setting of [3]. Similarly, if in these definitions we use δ_q-rings and δ_q-prolongation sequences *of not necessarily p-adically complete rings* instead, we obtain the concept of *δ_q-modular form* (over a δ_q-ring A); we then denote the corresponding spaces by $M_q^r(\Gamma_1(N), A, w)$ and $I_q^r(\Gamma_1(N), A, w)$, respectively. For $N = 1$, these δ_q-modular forms are the δ-modular forms in the geometric setting of [3].

As before, we can speak of forms in $M_p^r(\Gamma_1(N), A, w)$ and $I_p^r(\Gamma_1(N), A, w)$ that are essentially of level one. We denote them by

$$M_p^r\big(\Gamma_1(N), A, w\big)_1, \qquad I_p^r\big(\Gamma_1(N), A, w\big)_1,$$

and once again, these spaces are isomorphic to $M_p^r(A, w)$ and $I_p^r(A, w)$, respectively. We introduce a similar notation in the case of δ_q-modular forms.

There are natural injective maps

$$M_p^r\big(\Gamma_1(N), A, w\big) \to M_{pq}^r\big(\Gamma_1(N), A, w\big),$$
$$M_q^r\big(\Gamma_1(N), A, w\big) \to M_{pq}^r\big(\Gamma_1(N), A, w\big).$$

A similar statement holds for the I spaces in place of the M spaces above.

If in all of the definitions above we insist that elliptic curves have ordinary reduction mod p, then we get a new set of concepts that will be indicated using *ord* in the notation. So, for instance, instead of the sets $\mathbb{M}(A)$, $M_{pq}^r(A, w)$, $I_{pq}^r(A, w)$, etc., we will have sets $\mathbb{M}_{ord}(A)$, $M_{pq,ord}^r(A, w)$, $I_{pq,ord}^r(A, w)$, etc. Lemma 2.3 continues to hold for these "*ord*" sets. We also have natural maps $\mathbb{M}_{ord}(A) \to \mathbb{M}(A)$, $M_{pq}^r(A, w) \to M_{pq,ord}^r(A, w)$, $I_{pq}^r(A, w) \to I_{pq,ord}^r(A, w)$, etc.

Notice that

$$\mathbb{M}_{ord}(A) = \big\{(a, b) \in \mathbb{M}(A);\ E_{p-1}(a, b) \in A^\times\big\},$$

where $E_{p-1} \in \mathbb{Z}_p[a_4, a_6]$ is the polynomial whose Fourier series is the normalized Eisenstein series $E_{p-1}(q)$ of weight $p - 1$. Here *normalized* means that the constant coefficient is 1.

2.4. The main examples

Let A be a p-adically complete $\{\delta_p, \delta_q\}$-ring that is an integral Noetherian domain of characteristic zero, and let $N \geqslant 1$. The construction in [7, Remark 8.3], provides a recipe that attaches to any $\{\delta_p, \delta_q\}$-prolongation sequence S^* of Noetherian, p-adically complete rings over A, and to any pair (E, ω) consisting of an elliptic curve E/S^0 and an invertible 1-form ω on E, elements

$$f_p^1\big(E, \omega, S^*\big), f_q^1\big(E, \omega, S^*\big) \in S^1.$$

Moreover, in the case where E/S^0 has ordinary reduction mod p, we define (cf. [1, Construction 3.2 and Theorem 5.1]) an element

$$f_p^{\partial}\big(E,\omega,S^*\big)\in\big(S^1\big)^{\times}.$$

We shall sometimes drop S^* from the notation.

The rules f_p^1 and f_q^1 define isogeny covariant modular forms,

$$f_p^1\in I_p^1(A,-1-\phi),\qquad f_q^1\in I_q^1(A,-2),$$

cf. [3, Construction 4.1 and Construction 4.9]. In particular f_p^1 and f_q^1 give rise to elements of $I_{pq}^1(A,-1-\phi)$ and $I_{pq}^1(A,-2)$, respectively, and hence, to elements in $I_{pq,ord}^1(A,-1-\phi)$ and $I_{pq,ord}^1(A,-2)$, respectively, which we continue to denote by f_p^1 and f_q^1. Similarly f_p^{∂} defines an element

$$f_p^{\partial}\in I_{p,ord}^1(A,\phi-1),$$

hence an element of $I_{pq,ord}^1(A,\phi-1)$, which we still denote by f_p^{∂}.

Remark 2.4. Let $w_1,w_2\in W$ be two weights of order $\leqslant r$ such that $\deg(w_1)=\deg(w_2)\in 2\mathbb{Z}$. Then $w_2-w_1=(\phi-1)w$ for some $w\in W$, and the map

$$\begin{gathered}I_{pq,ord}^r(A,w_1)\to I_{pq,ord}^r(A,w_2),\\ g\mapsto\big(f_p^{\partial}\big)^w g\end{gathered}$$

is clearly an isomorphism. In particular, we may consider the forms

$$f_{p,-2}^{1\phi^i}:=\big(f_p^{\partial}\big)^{\frac{\phi^{i+1}+\phi^i-2}{\phi-1}}\big(f_p^1\big)^{\phi^i}\in I_{p,ord}^s(A,-2).$$

Similarly, we may consider the forms

$$f_{p,-1-\phi^s}^{1\phi^i}=\big(f_p^{\partial}\big)^{\frac{\phi^{i+1}+\phi^i-1-\phi^s}{\phi-1}}\big(f_p^1\big)^{\phi^i}\in I_{p,ord}^s\big(A,-1-\phi^s\big).$$

Theorem 2.5.

(1) *For any $s\geqslant 1$, the space $I_{p,ord}^s(A,-2)\otimes L$ has an L-basis consisting of the forms*

$$f_{p,-2}^1,f_{p,-2}^{1\phi},\dots,f_{p,-2}^{1\phi^{s-1}}.$$

(2) *The space $I_q^1(A,-2)\otimes L$ has a basis consisting of f_q^1.*

(3) *For any* $s \geqslant 1$, *let*

$$f_p^s := \sum_{i=0}^{s-1} p^{s-1-i} f_{p,-1-\phi^s}^{1\phi^i} \in I_{p,ord}^s\big(A, -1-\phi^s\big).$$

Then f_p^s *belongs to* $I_p^s(A, -1-\phi^s)$, *and it constitutes a basis for it.*

(4) *For any* $s \geqslant 1$, *the space* $I_{p,ord}^s(A, -2) \otimes L$ *has an* L*-basis consisting of the forms*

$$f_{p,-2}^1, f_{p,-2}^2, \ldots, f_{p,-2}^s.$$

Proof. (1) is implicit in [1]; it also follows directly by Proposition 8.75 in [6]. (2) is proved in [3, Corollary 7.24]. (3) is implicit in [1], and also follows by Proposition 8.61 and Theorem 8.83 in [6]. (4) is a clear consequence of (1) and (3). □

Remark 2.6. Let us recall that the construction in Remark 8.3 of [7] provides a recipe that attaches to any $\{\delta_p, \delta_q\}$-prolongation sequence S^* of Noetherian p-adically complete rings over A, and to any pair (E, ω) consisting of an elliptic curve E/S^0 and an invertible 1-form ω on E, elements

$$f_p^s\big(E, \omega, S^*\big), f_q^s\big(E, \omega, S^*\big) \in S^s$$

for all $s \geqslant 1$. (Again, we will usually drop S^* from notation.) The rules f_p^s and f_q^1 coincide with the forms introduced in Theorem 2.5, so they are, in particular, $\{\delta_p, \delta_q\}$-modular forms. However, notice that for $s \geqslant 2$, the rules f_q^s are not $\{\delta_p, \delta_q\}$-modular forms. For instance, for $s = 2$, we have the transformation law

$$f_q^2(E, \lambda\omega) = \lambda^2 f_q^2(E, \omega) + 2\lambda(\delta_q\lambda) f_q^1(E, \omega).$$

2.5. $\{\delta_p, \delta_q\}$-*Fourier expansions*

Assume, in the discussion below, that $N > 4$. We start by recalling the background of *classical* Fourier expansions; we will freely use the notation in [8, p. 112]. (The discussion in [8, p. 112], involves the model $X_\mu(N)$ instead of the model $X_1(N)$ used here, but these two models, and hence the two theories, are isomorphic over $\mathbb{Z}[1/N, \zeta_N]$, cf. [8, p. 113].)

There is a point $s_\infty : \mathbb{Z}[1/N, \zeta_N] \to X_1(N)_{\mathbb{Z}[1/N,\zeta_N]}$ arising from the generalized elliptic curve $\mathbb{P}^1_{\mathbb{Z}[1/N,\zeta_N]}$ with its canonical embedding of $\mu_{N,\mathbb{Z}[1/N,\zeta_N]} \simeq (\mathbb{Z}/N\mathbb{Z})_{\mathbb{Z}[1/N,\zeta_N]}$; the complex point corresponding to s_∞ is the cusp $\Gamma_1(N) \cdot \infty$. The map s_∞ is a closed immersion. We denote by $\tilde{X}_1(N)_{\mathbb{Z}[1/N,\zeta_N]}$ the completion of $X_1(N)_{\mathbb{Z}[1/N,\zeta_N]}$ along the image of s_∞.

Let us now consider an indeterminate $\mathbf{q}$ and the Tate generalized elliptic curve

$$\mathrm{Tate}(\mathbf{q})/\mathbb{Z}[1/N, \zeta_N][\![\mathbf{q}]\!],$$

defined as the projective closure of the plane affine curve given by

$$y^2 = x^3 - \frac{1}{48}E_4(\mathbf{q})x - \frac{1}{864}E_6(\mathbf{q}),$$

where E_4 and E_6 are the Eisenstein series

$$E_4(\mathbf{q}) = 1 + 240 \cdot s_3(\mathbf{q}),$$
$$E_6(\mathbf{q}) = 1 - 504 \cdot s_5(\mathbf{q}),$$

and s_m is defined by

$$s_m(\mathbf{q}) := \sum_{n \geqslant 1} \frac{n^m \mathbf{q}^n}{1 - \mathbf{q}^n} \in \mathbb{Z}[\![\mathbf{q}]\!].$$

(The normalization chosen here for the Tate curve is the same as that in [3], but slightly different from the usual one, for instance, that used in [7]; the Tate curve used in the present paper is, however, isomorphic over $\mathbb{Z}[\sqrt{-1}][\![\mathbf{q}]\!]$, and hence over $R[\![\mathbf{q}]\!]$, to the Tate curve in [7].) The curve $\mathrm{Tate}(\mathbf{q})$ is equipped with a canonical immersion

$$\alpha_{can} : \mu_{N, \mathbb{Z}[1/N, \zeta_N]} \simeq (\mathbb{Z}/N\mathbb{Z})_{\mathbb{Z}[1/N, \zeta_N]} \to \mathrm{Tate}(\mathbf{q}),$$

so there is an induced map $\operatorname{Spec} \mathbb{Z}[1/N, \zeta_N][\![\mathbf{q}]\!] \to X_1(N)_{\mathbb{Z}[1/N, \zeta_N]}$. There is an induced isomorphism

$$\operatorname{Spf} \mathbb{Z}[1/N, \zeta_N][\![\mathbf{q}]\!] \to \tilde{X}_1(N)_{\mathbb{Z}[1/N, \zeta_N]}. \tag{3}$$

There is also a canonical 1-form $\omega_{can} = dx/y$ on the elliptic curve $\mathrm{Tate}(\mathbf{q})$ over

$$\mathbb{Z}[1/N, \zeta_N]((\mathbf{q})) := \mathbb{Z}[1/N, \zeta_N][\![\mathbf{q}]\!][1/\mathbf{q}],$$

such that, for any classical modular form f on $\Gamma_1(N)$ over $\mathbb{C}$, the series

$$f_\infty := f_\infty(\mathbf{q}) := f\big(\mathrm{Tate}(\mathbf{q})/\mathbb{C}((\mathbf{q})), \alpha_{can}, \omega_{can}\big)$$

has image in $\mathbf{q}\mathbb{C}[\![\mathbf{q}]\!]$, and is the classical Fourier expansion at the cusp $\Gamma_1(N) \cdot \infty$. (Here we view f as a function of triples in the sense of [11].)

Let us now move onto the $\{\delta_p, \delta_q\}$-theory. We fix a prime p that does not divides N, a $\{\delta_p, \delta_q\}$-ring A that is an integral p-adically complete Noetherian domain of characteristic zero, and a homomorphism $\mathbb{Z}[1/N, \zeta_N] \to A$. Let $\mathbf{q}^{(i,j)}$ be indeterminates over A parameterized by non-negative integers i, j, such that $\mathbf{q}^{(0,0)} = \mathbf{q}$. We set $A((\mathbf{q})) := A[\![\mathbf{q}]\!][\mathbf{q}^{-1}]$, and

$$S^n_{pq,\infty} := A((\mathbf{q}))\widehat{\ }\big[\mathbf{q}^{(i,j)}\big|_{1 \leqslant i+j \leqslant n}\big]\widehat{\ }.$$

There is a unique structure of $\{\delta_p, \delta_q\}$-prolongation sequence on $S^n_{pq,\infty}$ over A that extends that of $A[\mathbf{q}^{(\leqslant r)}, \mathbf{q}^{-1}]\widehat{\ }$, and sends $A[\![\mathbf{q}]\!]$ into $A[\![\mathbf{q}]\!][\delta_p \mathbf{q}, \delta_q \mathbf{q}]\widehat{\ }$.

Finally, we define the $\{\delta_p, \delta_q\}$-*Fourier expansion map*

$$E_\infty : M^r_{pq}\big(\Gamma_1(N), A, w\big) \to S^r_{pq,\infty},$$
$$f \mapsto E_\infty(f) = f_\infty$$

by the formula

$$f_\infty := f_\infty\big(\mathbf{q}^{(i,j)}\big|_{0\leqslant i+j\leqslant r}\big) := f\big(\mathrm{Tate}(\mathbf{q})/S^0_{pq,\infty}, \alpha_{can}, \omega_{can}, S^*_{pq,\infty}\big).$$

By Lemma 2.3, we get an induced map

$$E_\infty : M^r_{pq}(A, w) \to S^r_{pq,\infty}.$$

Since the normalized Eisenstein series $E_{p-1}(q)$ is congruent to 1 mod p, we get $\{\delta_p, \delta_q\}$-Fourier expansion maps

$$\begin{aligned} E_\infty : M^r_{pq,ord}\big(\Gamma_1(N), A, w\big) &\to S^r_{pq,\infty}, \\ E_\infty : M^r_{pq,ord}(A, w) &\to S^r_{pq,\infty}. \end{aligned} \tag{4}$$

Proposition 2.7 *($\{\delta_p, \delta_q\}$-expansion principle). The $\{\delta_p, \delta_q\}$-Fourier expansion maps* (4) *are injective, with torsion free cokernel.*

Proof. This follows exactly as in [6, Proposition 8.29], where the Serre–Tate expansions (rather than the Fourier expansions) were considered. □

If instead of $\{\delta_p, \delta_q\}$ we use δ_p (or δ_q) only, we arrive at a δ_p-prolongation sequence

$$S^n_{p,\infty} := A((\mathbf{q}))\hat{}\big[\mathbf{q}^{(i,0)}:\ 1 \leqslant i \leqslant n\big]\hat{},$$

or at a δ_q-prolongation sequence

$$S^n_{q,\infty} := A((\mathbf{q}))\big[\mathbf{q}^{(0,j)}:\ 1 \leqslant j \leqslant n\big],$$

respectively. We have corresponding δ_p-Fourier or δ_q-Fourier expansion maps. These "ordinary" objects are compatible with the partial differential objects in an obvious way.

We introduce now certain subspaces of $S^r_{pq,\infty}$. For each prime l different from p, there is a unique morphism of $\{\delta_p, \delta_q\}$-prolongation sequences $\varphi^*_l : S^*_{pq,\infty} \to S^*_{pq,\infty}$ such that $\varphi^0_l(F(\mathbf{q})) = F(\mathbf{q}^l)$. Then, for each even integer m, we define the following A-submodules of $S^r_{pq,\infty}$:

$$I^r_{pq,\infty}(A, m) := \big\{ f \in S^r_{pq,\infty} \bigm| \varphi_l(f) = l^{-m/2} \cdot f \text{ for all } l \neq p \big\}.$$

Proposition 2.8. *Suppose that w is a weight of even degree m. Then the $\{\delta_p, \delta_q\}$-Fourier expansion maps* (4) *(see Proposition* 2.7*) send both, $I^r_{pq}(\Gamma_1(N), A, w)_1$ and $I^r_{pq,ord}(\Gamma_1(N), A, w)_1$, into $I^r_{pq,\infty}(A, m)$.*

Proof. We use the same argument as in the proof of Proposition 7.13 in [3]. □

2.6. Convection equations

In what follows, our purpose is to investigate the space $I^1_{pq,ord}(A,-2)$.

Consider the following two series in $S^1_{pq,\infty}$:

$$\Psi_p = \frac{1}{p}\cdot\log\left(1+p\frac{\delta_p\mathbf{q}}{\mathbf{q}^p}\right) = \sum_{n=1}^{\infty}(-1)^{n-1}p^{n-1}n^{-1}\left(\frac{\delta_p\mathbf{q}}{\mathbf{q}^p}\right)^n,$$

$$\Psi_q = \frac{\delta_q\mathbf{q}}{\mathbf{q}}.$$

Clearly we have $\Psi_p, \Psi_q \in I^1_{pq,\infty}(A,-2)$.

Let L be the fraction field of A.

Proposition 2.9. *The series Ψ_p, Ψ_q form an L-basis of $I^1_{pq,\infty}(A,-2)\otimes L$.*

Proof. Consider the map

$$c : I^1_{pq,\infty}(A,-2) \to A^2,$$
$$c(f) = \big(c_p(f), c_q(f)\big),$$

where $c_p(f)$ and $c_q(f)$ are the coefficients of $\frac{\delta_p\mathbf{q}}{\mathbf{q}^p}$ and $\frac{\delta_q\mathbf{q}}{\mathbf{q}}$ in f, respectively. It is enough to show that the map c in injective.

For any $i, j \geqslant 0$, let us set

$$\partial_{ij} = \frac{\partial^{i+j}}{(\partial\delta_p\mathbf{q})^i(\partial\delta_q\mathbf{q})^j},$$

and let $g = g(\mathbf{q}, \delta_p\mathbf{q}, \delta_q\mathbf{q}) \in I^1_{pq,\infty}(A,-2)$. Since $\delta_q(\mathbf{q}^2) = 2\mathbf{q}\delta_q\mathbf{q}$ and $\delta_p(\mathbf{q}^2) = 2\mathbf{q}^p\delta_p\mathbf{q} + p(\delta_p\mathbf{q})^2$, it follows that

$$g\big(\mathbf{q}^2, 2\mathbf{q}^p\delta_p\mathbf{q} + p(\delta_p\mathbf{q})^2, 2\mathbf{q}\delta_q\mathbf{q}\big) = \varphi_2(g) = 2g(\mathbf{q}, \delta_p\mathbf{q}, \delta_q\mathbf{q}).$$

We may easily check by induction that

$$(\partial_{ij}g)\big(\mathbf{q}^2, 2\mathbf{q}^p\delta_p\mathbf{q} + p(\delta_p\mathbf{q})^2, 2\mathbf{q}\delta_q\mathbf{q}\big)2^{i+j}\mathbf{q}^i\big(\mathbf{q}^p + p\delta_p\mathbf{q}\big)^j + U_{ij} = 2\cdot(\partial_{ij}g)(\mathbf{q}, \delta_p\mathbf{q}, \delta_q\mathbf{q}), \tag{5}$$

where U_{ij} is a linear combination with coefficients in the ring $S^1_{pq,\infty}$ of series of the form

$$(\partial_{i'j'}g)\big(\mathbf{q}^2, 2\mathbf{q}^p\delta_p\mathbf{q} + p(\delta_p\mathbf{q})^2, 2\mathbf{q}\delta_q\mathbf{q}\big)$$

with $i' + j' < i + j$. If we assume that g is in the kernel of the map c, setting $\delta_q\mathbf{q} = \delta_p\mathbf{q} = 0$ in (5), and using Lemma 7.22 in [3], we then obtain by induction that $(\partial_{ij}g)(\mathbf{q}, 0, 0) = 0$ for all i, j, and so g must be identically 0. $\square$.

In what follows, our $\{\delta_p, \delta_q\}$-ring A will be the ring $R = (\mathbb{Z}_p^{ur})\widehat{\ }$. We shall denote by $K = R[1/p]$ its fraction field.

Lemma 2.10. *There exist $c, \gamma \in \mathbb{Z}_p^\times$ such that the following hold:*

(1) *f_p^1 has $\{\delta_p, \delta_q\}$-Fourier expansion $c \cdot \Psi_p$.*
(2) *f_q^1 has $\{\delta_p, \delta_q\}$-Fourier expansion $\gamma \cdot \Psi_q$.*
(3) *f_p^∂ has $\{\delta_p, \delta_q\}$-Fourier expansion 1.*

Proof. Cf. [3, Corollary 7.26] and [1, Construction 3.2 and Theorem 5.1]. □

Let us recall the form $f_{p,-2}^1 = f_p^\partial f_p^1$ in Remark 2.4.

Theorem 2.11. *The forms f_q^1 and $f_{p,-2}^1$ form a K-basis of $I_{pq,ord}^1(R, -2) \otimes K$.*

In particular, any element $f \in I_{pq,ord}^1(R, -2)$ can be written as a K-linear combination

$$f = \varphi_q^1 + \varphi_p^1, \tag{6}$$

where $\varphi_q^1 \in K \cdot f_q^1$ and $\varphi_p^1 \in K \cdot f_{p,-2}^1$. Such a linear combination can be referred to as an *arithmetic convection equation*.

Proof. By Propositions 2.7 and 2.8, we have an injective map

$$I_{pq,ord}^1\big(\Gamma_1(N), R, -2\big)_1 \otimes K \to I_{pq,\infty}^1(R, -2) \otimes K.$$

The result follows by Lemma 2.10 and Proposition 2.9. □

If we combine the theorem above and the observations in Remark 2.4, we obtain bases for any of the spaces $I_{pq,ord}^1(R, w)$ with $\mathrm{ord}(w) \leqslant 1$ and $\deg(w) = -2$.

We investigate the solution spaces of equations defined by this type of forms in the next section.

2.7. Heat equations

Motivated by (6) and the notion introduce above, we now define an *arithmetic heat equation* to be one given by a form of the type

$$f = \varphi_q^1 + \varphi_p^2 \in I_{pq,ord}^2(R, -2), \tag{7}$$

where $\varphi_q^1 \in I_q^1(R, -2)$ and $\varphi_p^2 \in I_{p,ord}^2(R, -2)$. Thus, arithmetic convection equations are special cases of arithmetic heat equations, those where $\varphi_p^2 \in I_{p,ord}^1(R, -2)$.

We investigate their solution sets in the next section also.

2.8. Non-existence of wave equations

Motivated once again by (6) and now (7) also, we could try to introduce arithmetic analogues of wave equations by looking at forms of the type

$$f = \varphi_q^2 + \varphi_p^2 \in I^2_{pq,ord}(R, -2),$$

where $\varphi_q^2 \in I_q^2(R, -2)$ and $\varphi_p^2 \in I^2_{p,ord}(R, -2)$, and the corresponding equations they define. However, our next result shows that doing so does not lead to anything genuinely new.

Theorem 2.12. *We have that* $I_q^1(R, -2) = I_q^2(R, -2)$.

Remark 2.13. We expect that $I_q^1(R, -2) = I_q^r(R, -2)$ for all $r \geqslant 2$. However, the arguments that we will use to prove Theorem 2.12 seem to work for the case $r = 2$ only.

Remark 2.14. We expect that $I^2_{pq,ord}(R, -2) \otimes K$ has a basis consisting of the elements

$$f_q^1, \quad \left(f_p^\partial\right)^2\left(f_q^1\right)^\phi, \quad f_p^\partial f_p^1, \quad \left(f_p^\partial\right)^{\phi+2}\left(f_p^1\right)^\phi, \tag{8}$$

which would represent a strengthening of Theorem 2.12. Indeed, if this theorem were false, then $I_q^2(R, -2) \otimes K$ would have dimension $\geqslant 2$, and so two K-linearly independent elements of this space together with the last 3 of the 4 elements in (8) would yield 5 linearly independent elements in $I^2_{pq,ord}(R, -2) \otimes K$.

In order to prove Theorem 2.12, we now discuss some preliminary results. The strategy that we follow in this proof is inspired by a method of Barcau [1]. For convenience, we shall use the notation $x^{(r)}$ for $\delta_q^r x$, and when appropriate, the notation x' and x'' for $x^{(1)}$ and $x^{(2)}$, respectively.

First, we recall [3] that the space $M_q^r(R, m)$, $m \in \mathbb{Z}$, identifies with the space of all rational functions

$$f \in M_q^r := R\left[a_4, a_6, \ldots, a_4^{(r)}, a_6^{(r)}, \Delta^{-1}\right]$$

that have *weight* m, in the sense that

$$f\left(\ldots, \delta_q^i\left(\Lambda^4 a_4\right), \delta_q^i\left(\Lambda^6 a_6\right), \ldots, \Lambda^{12}\Delta^{-1}\right) = \Lambda^m f\left(\ldots, a_4^{(i)}, a_6^{(i)}, \ldots, \Delta^{-1}\right) \tag{9}$$

in the ring

$$R\left[a_4, a_6, \Lambda, \ldots, a_4^{(r)}, a_6^{(r)}, \Lambda^{(r)}, \Lambda^{-1}, \Delta^{-1}\right].$$

We recall also [3] that we have natural maps — called the $(\mathbf{q}, \ldots, \mathbf{q}^{(r)})$-Fourier expansion maps:

$$\begin{aligned} M_q^r &\to R((\mathbf{q}))\left[\mathbf{q}', \ldots, \mathbf{q}^{(r)}\right], \\ u &\mapsto u_\infty. \end{aligned} \tag{10}$$

We have the following:

Lemma 2.15. *The* $(\mathbf{q}, \mathbf{q}')$*-Fourier expansion map* $M_q^1 \to R((\mathbf{q}))[\mathbf{q}']$ *is injective.*

Proof. This is contained in [3, Proposition 7.10]. □

Remark 2.16. Notice that the map in (10) is not injective for $r \geqslant 2$; cf. [3, Proposition 7.10].

On the ring M_q^r, we introduce the following derivation operators:

$$\partial_r = 16a_4^2 \frac{\partial}{\partial a_6^{(r)}} - 72a_6 \frac{\partial}{\partial a_4^{(r)}},$$

$$D_r = 4a_4 \frac{\partial}{\partial a_4^{(r)}} + 6a_6 \frac{\partial}{\partial a_6^{(r)}}.$$

Notice that ∂_0 is the usual Serre operator (see [11, Appendix]), while D_0 is the usual Euler operator for weighted homogeneous polynomials. In general, these operators are "geometric" analogues of certain "arithmetic" operators introduced in [1].

Lemma 2.17. *If* $f \in M_q^r$ *has weight* m*, then* $\partial_r f$ *has weight* $m+2$.

Proof. Apply the operator ∂_r in (9). □

Lemma 2.18. *If* $f \in M_q^r$ *has weight* m *and* $r \geqslant 1$*, then* $D_r f = 0$.

Proof. Apply the operator $\frac{\partial}{\partial \Lambda^{(r)}}$ in (9), and set $\Lambda = 1$. □

Lemma 2.19. *If* $f \in M_q^r$ *has weight* m *and* $r \geqslant 1$*, then we have the equality of* δ_q*-Fourier series*

$$(\partial_r f)_\infty = 12\mathbf{q} \frac{\partial f_\infty}{\partial \mathbf{q}^{(r)}}.$$

Proof. Let us write $(a_4)_\infty = a_4(\mathbf{q})$, with a similar notation for a_6. We have that

$$12\mathbf{q} \frac{da_4(\mathbf{q})}{d\mathbf{q}} = 4P(\mathbf{q})a_4(\mathbf{q}) - 72a_6(\mathbf{q}),$$

$$12\mathbf{q} \frac{da_6(\mathbf{q})}{d\mathbf{q}} = 6P(\mathbf{q})a_6(\mathbf{q}) + 16a_4^2(\mathbf{q}), \tag{11}$$

where $P(\mathbf{q}) \in R[\![\mathbf{q}]\!]$ is the Ramanujan series [11, Appendix]. Consequently, if V denotes the vector

$$\big(\ldots, \delta_q^i\big(a_4(\mathbf{q})\big), \delta_q^i\big(a_6(\mathbf{q})\big), \ldots\big),$$

we have that

$$\begin{aligned}
12\mathbf{q}\frac{\partial f_\infty}{\partial \mathbf{q}^{(r)}} &= 12\mathbf{q}\left[\frac{\partial f}{\partial a_4^{(r)}}(V)\frac{\partial}{\partial \mathbf{q}^{(r)}}\big(\delta_q^r\big(a_4(\mathbf{q})\big)\big) + \frac{\partial f}{\partial a_6^{(r)}}(V)\frac{\partial}{\partial \mathbf{q}^{(r)}}\big(\delta_q^r\big(a_6(\mathbf{q})\big)\big)\right] \\
&= 12\mathbf{q}\left[\frac{\partial f}{\partial a_4^{(r)}}(V)\frac{da_4(\mathbf{q})}{d\mathbf{q}} + \frac{\partial f}{\partial a_6^{(r)}}(V)\frac{da_6(\mathbf{q})}{d\mathbf{q}}\right] \\
&= \big(4P(\mathbf{q})a_4(\mathbf{q}) - 72a_6(\mathbf{q})\big)\frac{\partial f}{\partial a_4^{(r)}}(V) + \big(6P(\mathbf{q})a_6(\mathbf{q}) + 16a_4^2(\mathbf{q})\big)\frac{\partial f}{\partial a_6^{(r)}}(V) \\
&= P(\mathbf{q})(D_r f)_\infty + (\partial_r f)_\infty.
\end{aligned}$$

The desired result now follows by Lemma 2.18. □

Lemma 2.20. *We have that $\delta_q f_q^1 \notin M_q^2(R,-2)$.*

Proof. Let us assume that the opposite is true, that is to say, that

$$\big(\delta_q f_q^1\big)\big(E, \lambda\omega, S^*\big) = \lambda^{-2}\big(\delta_q f_q^1\big)\big(E, \omega, S^*\big).$$

We apply δ_q in the identity

$$f_q^1\big(E, \lambda\omega, S^*\big) = \lambda^{-2} f_q^1\big(E, \omega, S^*\big),$$

and use the assumption to get that

$$\lambda(\delta_q\lambda) f_q^1\big(E, \omega, S^*\big) = 0,$$

which is, of course, false for the generic (E,ω) and for any λ with $\delta_q\lambda \neq 0$. □

We will also need the following explicit formula for f_q^1 that is due to Hurlburt [9]; cf. also [3, Proposition 4.10 and Corollary 7.24].

Lemma 2.21. *(See [9].) We have $f_q^1 = \gamma_1 \frac{2a_4a_6' - 3a_6a_4'}{\Delta}$, where $\gamma_1 \in R^\times$.*

(Of course the constant γ_1 above and the constant γ in Lemma 2.10 are explicitly related to each other, but this relation is irrelevant to us here.)

Proof of Theorem 2.12. Let $g \in I_q^2(R,-2)$. We want to show that $g \in I_q^1(R,-2)$. By [3, Propositions 7.13 and 7.23], we may assume that

$$g_\infty = \lambda\frac{\mathbf{q}'}{\mathbf{q}} + \mu\left(\frac{\mathbf{q}}{\mathbf{q}}\right)', \tag{12}$$

with $\lambda, \mu \in R$. Replacing g by g plus a multiple of f_q^1, we may assume that $\lambda = 0$. We will prove that $g = 0$.

By Lemma 2.19, we have that

$$(\partial_2 g)_\infty = 12\mathbf{q}\frac{\partial g_\infty}{\partial \mathbf{q}''} = 12\mu.$$

By Lemma 2.17, we see that $\partial_2 g$ has weight 0. Now, the constant $12\mu \in M_q^2$ has weight 0 also, and δ_q-Fourier expansion $(12\mu)_\infty = 12\mu$. By Proposition 2.7, it follows that

$$\partial_2 g = 12\mu. \tag{13}$$

Notice that by Lemma 2.18 we have that

$$D_2 g = 0. \tag{14}$$

By Lemma 2.21, we have

$$\delta_q f_q^1 \in \gamma_1 \frac{2a_4 a_6'' - 3a_6 a_4''}{\Delta} + M_q^1.$$

A trivial computation yields

$$\partial_2 \delta_q f_q^1 = \gamma_1/2, \qquad D_2 \delta_q f_q^1 = 0. \tag{15}$$

We set $h := g + \frac{24\mu}{\gamma_1}\delta_q f_q^1 \in M_q^2$. By (13), (14) and (15), we obtain that

$$\partial_2 h = 0, \qquad D_2 h = 0. \tag{16}$$

Also, by (12), h has $(\mathbf{q}, \mathbf{q}')$-Fourier expansion

$$h_\infty = \gamma_2 \left(\frac{\mathbf{q}'}{\mathbf{q}}\right)', \qquad \gamma_2 = \mu + \frac{24\mu\gamma}{\gamma_1}. \tag{17}$$

Now, by (16), we get that

$$\frac{\partial h}{\partial a_4''} = \frac{\partial h}{\partial a_6''} = 0,$$

hence $h \in M_q^1$. On the other hand, the element $\frac{\gamma_2}{\gamma}\delta_q f_q^1 \in M_q^1$ has $(\mathbf{q}, \mathbf{q}')$-Fourier expansion

$$\left(\frac{\gamma_2}{\gamma}\delta_q f_q^1\right)_\infty = \gamma_2 \left(\frac{\mathbf{q}'}{\mathbf{q}}\right)'. \tag{18}$$

By Lemma 2.15 and (17) and (18), we must have that $h = \frac{\gamma_2}{\gamma}\delta_q f_q^1$, and therefore, $g = \gamma_3 \delta_q f_q^1$, $\gamma_3 \in R$. By Lemma 2.20, we conclude that $\gamma_3 = 0$, hence $g = 0$, and we are done. $\square$

3. Solutions

We now analyze the sets of solutions of the equations introduced in the previous section. We first examine solutions with bad reduction at $q=0$; then we shall examine solutions with good reduction at $q=0$.

3.1. Bad reduction

We define the following subset of $\mathbb{M}(R((q))\widehat{\ })$:

$$\mathbb{M}\big(R((q))\widehat{\ }\big)_{bad} := \big\{(a,b) \in R[\![q]\!]^{\times} \times R[\![q]\!]^{\times};\ 4a^3+27b^2 \in qR[\![q]\!]^{\times}\big\} \subset \mathbb{M}\big(R((q))\widehat{\ }\big).$$

We will prove the following

Lemma 3.1. *We have that* $\mathbb{M}(R((q))\widehat{\ })_{bad} \subset \mathbb{M}_{ord}(R((q))\widehat{\ })$.

We assume this lemma for the time being. For any $f \in M^r_{pq,ord}(R,-2)$, we define the set of *bad reduction solutions* of the equation $f=0$ by

$$\mathcal{U}_{bad} := \big\{(a,b) \in \mathbb{M}\big(R((q))\widehat{\ }\big)_{bad};\ f(a,b)=0\big\}.$$

We aim at the description of this space for arithmetic heat equations

$$f := f^1_q + \varphi^2_p;$$

cf. (7).

By Theorem 2.5,

$$\varphi^2_p = \lambda f^1_{p,-2} + \epsilon f^{1\phi}_{p,-2},$$

where $\lambda, \epsilon \in K$. In studying the equation $f=0$, we may assume that $\lambda, \epsilon \in R$. We will further assume that $\lambda \in R^{\times}$. Let

$$\int : K[\![q]\!] \to qK[\![q]\!]$$

be the usual integration operator. Also consider the Fourier expansions of a_4, a_6 respectively:

$$a_{4,\infty}(\mathbf{q}) := -\frac{1}{48}E_4(\mathbf{q}), \qquad a_{6,\infty}(\mathbf{q}) := -\frac{1}{864}E_6(\mathbf{q}) \in R[\![\mathbf{q}]\!].$$

For $n \geqslant -1$, we define inductively the rational functions

$$b_n = b_n(x,y) \in \mathbb{Z}\left[x, y, \frac{1}{(1-y)(1-y^2)\cdots(1-y^n)}\right]$$

by setting $b_{-1}=0$, $b_0=1$, and the recurrence relation

$$b_n = \frac{1-x}{1-y^n} b_{n-1} + \frac{x}{1-y^n} b_{n-2}. \tag{19}$$

Notice that we have

$$b_n(x,0) = \frac{1+(-1)^n x^{n+1}}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n,$$
$$b_n(0,y) = \frac{1}{(1-y)(1-y^2)\cdots(1-y^n)}.$$

Both formulae above have, of course, a quantum theoretic flavor.

For $0 \neq \kappa \in \mathbb{Z}_+$, $z \in \mathbb{Q}_p$, $\alpha \in K$, $\eta \in K^\times$ and $v \in K[\![q]\!]^\times$, let us define

$$\begin{aligned}
u^z_{a,\kappa,\alpha} &:= \sum_{n \geqslant 0} b_n(pz,p) \alpha^{\phi^n} q^{\kappa p^n} \in K[\![q]\!],\\
u^z_{m,\kappa,\alpha} &:= \exp\left(\int u^z_{a,\kappa,\alpha} \frac{dq}{q}\right) \in 1 + qK[\![q]\!],\\
u^z_{\Gamma,\eta,v} &:= \big(v^4 a_{4,\infty}(\eta q), v^6 a_{6,\infty}(\eta q)\big) \in \mathbb{M}\big(K(\!(q)\!)\big),\\
u^z_{\Gamma,\eta,v,\kappa,\alpha} &:= \big(v^4 a_{4,\infty}\big(\eta q u^z_{m,\kappa,\alpha}\big), v^6 a_{6,\infty}\big(\eta q u^z_{m,\kappa,\alpha}\big)\big) \in \mathbb{M}\big(K(\!(q)\!)\big).
\end{aligned} \tag{20}$$

Here the indices a, m, Γ stand for the additive group $\mathbb{G}_a$, the multiplicative group $\mathbb{G}_m$, and the modular scheme $\mathbb{M}_\Gamma$ attached to $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. As observed in [7], the series

$$u^0_{m,\kappa,\alpha} = \exp\left(\frac{\alpha q^\kappa}{\kappa} + \sum_{n \geqslant 1} \frac{\alpha^{\phi^n} q^{\kappa p^n}}{\kappa p^n (1-p)(1-p^2)\cdots(1-p^n)}\right)$$

is a sort of hybrid between the Artin–Hasse exponential [12, p. 138] and a quantum exponential [10, p. 30].

Lemma 3.2. *If $\kappa \notin p\mathbb{Z}$, $z \in \mathbb{Z}_p$, and $\alpha \in R$, then*

$$u^z_{a,\kappa,\alpha} \in R[\![q]\!], \qquad u^z_{m,\kappa,\alpha} \in 1 + qR[\![q]\!].$$

If, in addition, $\eta \in R^\times$ and $v \in R[\![q]\!]^\times$, then

$$u^z_{\Gamma,\eta,v}, u^z_{\Gamma,\eta,v,\kappa,\alpha} \in \mathbb{M}\big(R(\!(q)\!)\big).$$

Proof. We freely use the theory in [7]. Let us consider the $\{\delta_p, \delta_q\}$-character of $\mathbb{G}_m$ defined by

$$\psi_m := \psi_q + (\kappa + \kappa z \phi)\psi_p,$$

where ψ_q, ψ_p are defined by the series Ψ_q, Ψ_p, respectively; cf. [7, Definition 7.3 and Example 7.4]. Using the terminology in [7], we have that ψ_m is non-degenerate, and its characteristic polynomial is

$$\mu(\xi_p, \xi_q) = \xi_q + p\kappa z\xi_p^2 + (\kappa - p\kappa z)\xi_p - \kappa.$$

Thus, μ has characteristic integer κ. By [7, Definitions 7.3 and 6.2], the basic series of ψ_m are equal to the series $u^z_{m,\kappa,\alpha}$. By [7, Lemma 7.6], we have that $u^z_{\kappa,\alpha} \in 1 + qR[\![q]\!]$. The remaining portion of the lemma is simple. $\square$

Remark 3.3. Still using the notation above, we recall from [7, Theorem 1.10], that the set of solutions in $1 + qR[\![q]\!]$ of $\psi_m = 0$ equals $\{u^z_{m,\kappa,\alpha} \mid \alpha \in R\}$.

Theorem 3.4. *Consider the arithmetic heat equation given by*

$$f := f^1_q + \lambda f^1_{p,-2} + \epsilon f^{1\phi}_{p,-2},$$

where $\lambda \in R^\times$, $\epsilon \in pR$, *and* $z := \frac{\epsilon}{\lambda} \in p\mathbb{Z}_p$. *Let* $\kappa := \frac{c}{\gamma} \cdot \lambda \in R^\times$, *and* $\beta \in R^\times$ *be such that*

$$\Psi_p(\beta) + z\Psi_p(\beta)^\phi = -\frac{1}{\kappa}$$

(such a β *always exists). If* $\mathcal{U}_{bad}$ *is the set of bad reduction solutions of the equation* $f = 0$, *then the following hold:*

(1) *Assume* $\kappa \notin \mathbb{Z}_+$. *Then*

$$\mathcal{U}_{bad} = \{u^z_{\Gamma,\zeta\beta,v} \mid \zeta \in \mu(R),\ v \in R[\![q]\!]^\times\}.$$

(2) *Assume* $\kappa \in \mathbb{Z}_+$. *Then*

$$\mathcal{U}_{bad} = \{u^z_{\Gamma,\zeta\beta,v,\kappa,\alpha} \mid \zeta \in \mu(R),\ v \in R[\![q]\!]^\times,\ \alpha \in R\}.$$

We may view this as a "quantization" result: $\mathcal{U}_{bad}$ has jumps exactly at the integral positive values of κ.

Morally speaking, this theorem is about arithmetic heat equations that are close to, or coincide with, a given arithmetic convection equation. The case of an arithmetic convection equation corresponds to $\epsilon = 0$; the case $0 \neq \epsilon \in pR$ corresponds to arithmetic heat equations different that yield in the limit the arithmetic convection equation. The problem of how solutions behave when $\epsilon \to 0$ will be addressed in the next section. It is worth mentioning that heat equations of the form $f^1_q + \lambda f^2_{p,-2} = 0$, say, are not covered by our theorem; such equations are not "close to arithmetic convection equations."

We derive a preliminary result before the proof. We will use it to prove Lemma 3.1, which in turn, shall be used in the proof of the theorem.

Let us introduce the map

$$\begin{aligned} \iota : R[\![q]\!]^\times \times \big(R[\![q]\!]^\times/\{\pm 1\}\big) &\to \mathbb{M}\big(R((q))\widehat{\ }\big), \\ (u, v) &\mapsto \big(v^4 a_{4,\infty}(uq), v^6 a_{6,\infty}(uq)\big). \end{aligned} \tag{21}$$

Lemma 3.5. *The map* ι *in* (21) *is injective, and its image is* $\mathbb{M}(R((q))\widehat{\ })_{bad}$.

Proof. Let us assume that $\iota(u_1, v_1) = \iota(u_2, v_2)$. Then $j_\infty(u_1 q) = j_\infty(u_2 q)$, where

$$j_\infty(\mathbf{q}) = 1/\mathbf{q} + 744 + \cdots$$

is the Fourier expansion of the j-invariant. It follows that $\frac{1}{j_\infty}(u_1 q) = \frac{1}{j_\infty}(u_2 q)$. Since $\frac{1}{j_\infty(\mathbf{q})} = \mathbf{q} + \cdots$ has a compositional inverse, it follows that $u_1 q = u_2 q$, hence $u_1 = u_2$ and $v_1^2 = v_2^2$. This proves the injectivity of ι.

Let us now verify the identity of sets $\operatorname{Im}\iota = \mathbb{M}(R((q))\widehat{\ })_{bad}$. That the first set is included in the second is clear. Conversely, let $(a, b) \in \mathbb{M}(R((q))\widehat{\ })_{bad}$. Then

$$\frac{4a^3 + 27b^2}{a^3} \in qR[\![q]\!]^\times.$$

We let $\sigma \in qR[\![q]\!]^\times$ be the compositional inverse of $\frac{1}{j_\infty(\mathbf{q})} \in \mathbf{q}R[\![\mathbf{q}]\!]^\times$, and set

$$u = \frac{1}{q}\sigma\left(2^8 3^3 \frac{4a^3 + 27b^2}{a^3}\right) \in R[\![q]\!]^\times.$$

Then the elliptic curves $y^2 = x^3 + ax + b$ and $y^2 = x^3 + a_{4,\infty}(uq)x + a_{6,\infty}(uq)$ have the same j-invariant, and therefore, there exists an element v in the algebraic closure of the fraction field of $R((q))$ such that $a = v^4 a_{4,\infty}(uq)$ and $b = v^6 a_{6,\infty}(uq)$, respectively. We get that $v^2 \in R[\![q]\!]^\times$, and consequently, by Hensel, that $v \in R[\![q]\!]^\times$. So $(a, b) \in \operatorname{Im}\iota$, which proves the desired inclusion. $\square$

Proof of Lemma 3.1. By Lemma 3.5, any element in $\mathbb{M}(R((q))\widehat{\ })_{bad}$ is of the form $\epsilon(u, v)$. Let $E_{p-1} \in \mathbb{Z}_p[a_4, a_6]$ be the polynomial whose Fourier expansion is the Eisenstein series $E_{p-1}(q)$. Then

$$E_{p-1}\big(a_{4,\infty}(uq), a_{6,\infty}(uq)\big) = E_{p-1}(uq) \equiv 1 \mod p,$$

so $\epsilon(u, v)$ corresponds to an ordinary elliptic curve. $\square$

Proof of Theorem 3.4. If we compose a map $g : \mathbb{M}_{ord}(R((q))\widehat{\ }) \to R((q))\widehat{\ }$ with the map ι in (21), we obtain a map

$$g_\iota = g \circ \iota : R[\![q]\!]^\times \times \big(R[\![q]\!]^\times/\{\pm 1\}\big) \to R((q))\widehat{\ }.$$

By Lemma 3.5, ι induces a bijection between $\mathcal{U}_{bad}$ and the set

$$f_\iota^{-1}(0) \subset R[\![q]\!]^\times \times \big(R[\![q]\!]^\times/\{\pm 1\}\big).$$

For any $s \in R((q))\widehat{\ }$, and abusing notation a bit, we write $\Psi_p(s)$ in place of $\Psi_p(s, \delta_p s)$, and similarly for Ψ_q. Notice that

$$\Psi_p(s_1 s_2) = \Psi_p(s_1) + \Psi_p(s_2),$$

with a similar expression for Ψ_q instead. By Lemma 2.10, we have that

$$\begin{aligned}
\big(f_p^1\big)_\iota(u,v) &= c\cdot v^{-1-\phi}\cdot \Psi_p(uq) = c\cdot v^{-1-\phi}\cdot\big(\Psi_p(u)+\Psi_p(q)\big) = c\cdot v^{-1-\phi}\cdot\Psi_p(u),\\
\big(f_q^1\big)_\iota(u,v) &= \gamma\cdot v^{-2}\cdot\Psi_q(uq) = \gamma\cdot v^{-2}\cdot\big(\Psi_q(u)+\Psi_q(q)\big) = \gamma\cdot v^{-2}\cdot\big(\Psi_q(u)+1\big),\\
\big(f_p^\partial\big)_\iota(u,v) &= v^{\phi-1},
\end{aligned}$$

so we have that

$$f_\iota(u,v)=0$$

if, and only if,

$$\psi_m(u) := \Psi_q(u)+\kappa\Psi_p(u)+\kappa z\phi\Psi_p(u) = -1.$$

The latter has the obvious solution $u=\beta$, and any other solution in $R[\![q]\!]^\times$ is of the form $u=\beta\cdot u_+$, where $u_+\in R[\![q]\!]$ satisfies the homogeneous equation

$$\Psi_m(u_+)=0. \tag{22}$$

Clearly $u_0 := u_+(0)$ satisfies the equation

$$\Psi_p(u_0)+z\phi\Psi_p(u_0)=0. \tag{23}$$

This is equivalent to the condition $\Psi_p(u_0)=0$ (because if $\Psi_p(u_0)\neq 0$ then the terms in the left-hand side of (23) have distinct valuations). So $u_0=\zeta\in\mu(R)$ is a root of unity, and $u_+=\zeta u_1$, $u_1\in 1+qR[\![q]\!]$, $\Psi_m(u_1)=0$. By the proof of Lemma 3.2 and Remark 3.3, if $\kappa\in\mathbb{Z}$ then we must have $u_1=u^z_{m,\kappa,\alpha}$ for some $\alpha\in R$. On the other hand, if $\kappa\notin\mathbb{Z}_+$ then, by [7, Theorem 7.10], $u_1=1$. This finishes the proof. □

Remark 3.6. Notice that, by Lemma 3.5, the maps

$$\begin{aligned}
(\zeta,v) &\mapsto u_{\Gamma,\zeta\beta,v},\\
(\zeta,v,\alpha) &\mapsto u_{\Gamma,\zeta\beta,v,\kappa,\alpha}
\end{aligned}$$

are injective. Notice also that we may write

$$\mathcal{U}_{bad} = \bigcup_{\zeta\in\mu(R)} \mathcal{U}_\zeta, \tag{24}$$

where

$$\mathcal{U}_\zeta = \big\{u_{\Gamma,\zeta\beta,v} \bigm| v\in R[\![q]\!]^\times\big\}$$

or

$$\mathcal{U}_\zeta = \{ u_{\Gamma,\zeta\beta,v,\kappa,\alpha} \mid v \in R[\![q]\!]^\times,\ \alpha \in R \},$$

if $\kappa \notin \mathbb{Z}_+$ or $\kappa \in \mathbb{Z}_+$, respectively. According to these two cases, we have corresponding canonical identifications

$$\mathcal{U}_\zeta / R[\![q]\!]^\times \simeq 0 \quad \text{or} \quad \mathcal{U}_\zeta / R[\![q]\!]^\times \simeq R, \tag{25}$$

where the latter of the two is gotten via the α parameter. This picture will have an analogue in the case of good reduction, where $\mu(R) \simeq k^\times$ will be replaced by $k \setminus \{0, 1728, ss\}$, $k = R/pR$ viewed as the j-line parameterizing elliptic curves over k, and ss the supersingular points.

3.2. Good reduction

The analysis of the set of bad solutions has an analogue in the case of good reduction, which we now explore.

Let $(A, B) \in \mathbb{M}(R)$. We say that (A, B) is a *CL-point* (a canonical lift point) if the elliptic curve $y^2 = x^3 + Ax + B$ has ordinary reduction mod p, and is isomorphic over R to the canonical lift of its reduction mod p. We denote by $\mathbb{M}(R)_{CL} \subset \mathbb{M}_{ord}(R)$ the set of CL points. Also, let $f = f^1_q + \lambda f^1_{p,-2} + \epsilon f^{1\phi}_{p,-2}$, with $\lambda \in R^\times$, $\epsilon \in pR$, and set

$$\begin{aligned} \mathcal{U}_{ord} &= \{ (a, b) \in \mathbb{M}_{ord}(R[\![q]\!]) \mid f(a, b) = 0 \}, \\ \mathcal{U}_0 &= \{ (A, B) \in \mathbb{M}_{ord}(R) \mid f(A, B) = 0 \} \subset \mathcal{U}_{ord}. \end{aligned}$$

We refer to $\mathcal{U}_0$ as the set of *stationary solutions* of f.

Lemma 3.7. *We have that:*

(1) $\mathcal{U}_0 = \mathbb{M}(R)_{CL}$.
(2) *The map*

$$\begin{aligned} \mathbb{M}_{ord}(R[\![q]\!]) &\to \mathbb{M}_{ord}(R), \\ (a, b) &\mapsto (A, B) = (a(0), b(0)) \end{aligned}$$

induced by the map

$$q \mapsto 0,$$

sends $\mathcal{U}_{ord}$ onto $\mathcal{U}_0$.

Proof. Let us check (1). By Lemma 2.21, f^1_q vanishes on $\mathbb{M}(R)$. Now, a pair $(A, B) \in \mathbb{M}(R)$ belongs to $\mathbb{M}(R)_{CL}$ if, and only if, $f^1_p(A, B) = 0$; cf. [6, Propositions 7.18 and 8.89]. On the other hand, $f^1_p(A, B) = 0$ if, and only if,

$$\lambda f^\partial_p(A, B) f^1_p(A, B) + \epsilon f^\partial_p(A, B)^{\phi+2} f^1_p(A, B)^\phi = 0$$

(because if $f^1_p(A, B) \neq 0$, then the terms in the sum above have distinct valuations).

For (2), let $(a,b)\in\mathcal{U}_{ord}$, and $(A,B)=(a(0),b(0))$. We have that

$$0=f_q^1(a,b)_{|q=0}+\lambda f_p^\partial(a,b)_{|q=0}f_p^1(a,b)_{|q=0}+\epsilon f_p^\partial(a,b)_{|q=0}^{\phi+2}f_p^1(a,b)_{|q=0}^{\phi}.$$

By Lemma 2.21,

$$f_q^1=\gamma_1\frac{2aq\frac{db}{dq}-3bq\frac{da}{dq}}{\Delta(a,b)}\in qR[\![q]\!],$$

so $f_q^1(a,b)_{|q=0}=0$. Also $f_p^1(a,b)_{|q=0}=f_p^1(A,B)$, and $f_p^\partial(a,b)_{|q=0}=f_p^\partial(A,B)$ by the compatibility of f_p^1 and f_p^∂ with the δ_p-ring homomorphism

$$\begin{aligned}R[\![q]\!]&\to R,\\ q&\mapsto 0.\end{aligned}$$

Thus, $f(A,B)=0$, and the map $q\mapsto 0$ induces a map $\mathcal{U}_{ord}\to\mathcal{U}_0$ that is the identity on $\mathcal{U}_0$, therefore, surjective. $\square$

In the sequel, we extend the Serre–Tate expansion of modular forms [13] and δ_p-modular forms [4,5] to $\{\delta_p,\delta_q\}$-modular forms. Let $(A_0,B_0)\in\mathbb{M}_{ord}(k)$, $k=R/pR$, and E_0 be the (ordinary) elliptic curve $y^2=x^3+A_0x+B_0$. We fix a $\mathbb{Z}_p$-basis of the physical Tate module $T_p(E_0)$. Then, using the formal universal deformation space $R[\![t]\!]$ of E_0, the universal elliptic curve E_{def} over it, and the natural 1-form ω_{def} on it (cf. [4, pp. 212–213]), we can define a $\{\delta_p,\delta_q\}$-*Serre–Tate expansion map*

$$\begin{aligned}M_{pq}^n&\to S_{pq,def}^n:=R[\![t]\!]\big[\delta_p^i\delta_q^jt;\ 1\leqslant i+j\leqslant n\big]\widehat{\ },\\ f&\mapsto f_{E_0}:=f\big(E_{def},\omega_{def},S_{pq,def}^*\big).\end{aligned}$$

Remark 3.8. A change of basis in the Tate module $T_p(E_0)$ (corresponding to an element $\lambda\in\mathbb{Z}_p^\times$) has the effect of composing the map $f\mapsto f_{E_0}$ with the unique automorphism of prolongation sequences of $S_{pq,def}^n$ that sends t into $[\lambda^2](t)$; the latter series is the multiplication by λ^2 in the formal group of $\mathbb{G}_m$.

For the next result, we consider the following series:

$$\begin{aligned}\Psi_q(1+t)&=\frac{\delta_qt}{1+t}\in R[\![t]\!][\delta_qt],\\ \Psi_p(1+t)&=\frac{1}{p}\log\frac{\phi(1+t)}{(1+t)^p}=\frac{1}{p}\log\left(1+p\frac{\delta_pt+\frac{1+t^p-(1+t)^p}{p}}{(1+t)^p}\right)\in R[\![t]\!][\delta_pt]\widehat{\ }.\end{aligned}$$

Lemma 3.9. *There exist $c_{E_0},\gamma_{E_0}\in R^\times$ such that the following hold:*

(1) *f_p^1 has $\{\delta_p,\delta_q\}$-Serre–Tate expansion $c_{E_0}\cdot\Psi_p(1+t)$.*
(2) *f_q^1 has $\{\delta_p,\delta_q\}$-Serre–Tate expansion $\gamma_{E_0}\cdot\Psi_q(1+t)$.*
(3) *f_p^∂ has $\{\delta_p,\delta_q\}$-Serre–Tate expansion 1.*

Proof. (1) was proved in [4, Lemma 2.4]. (2) follows exactly as in [4, pp. 230–234]. (3) was proved in [5, Proposition 7.2]. $\square$

Remark 3.10. It is not clear what the relation between the constants c_{E_0} and γ_{E_0} in the lemma above and the constants c and γ in Lemma 2.10 is.

Remark 3.11. Let $a_{4,E_0}, a_{6,E_0} \in R[\![t]\!]$ be the Serre–Tate expansions of a_4 and a_6, respectively. Since the specialization

$$\begin{gathered} R[\![t]\!] \to R, \\ t \mapsto 0 \end{gathered}$$

sends E_{def} into an elliptic curve E/R that is isomorphic to the canonical lift of E_0/k, we get that $(a_{4,E_0}(0), a_{6,E_0}(0)) \in \mathbb{M}(R)_{CL}$.

Let us consider the sets

$$\mathbb{M}\big(R[\![q]\!]\big)_{CL,E_0} \subset \mathbb{M}\big(R[\![q]\!]\big)_{E_0} \subset \mathbb{M}_{ord}\big(R[\![q]\!]\big),$$

defined as follows:

(1) $\mathbb{M}(R[\![q]\!])_{E_0}$ consists of all $(a,b) \in \mathbb{M}(R[\![q]\!])$ such that the reduction mod p of the elliptic curve defined by $y^2 = x^3 + a(0)x + b(0)$ is isomorphic over k to E_0.
(2) $\mathbb{M}(R[\![q]\!])_{CL,E_0}$ consists of all $(a,b) \in \mathbb{M}(R[\![q]\!])$ such that the elliptic curve defined by $y^2 = x^3 + a(0)x + b(0)$ is isomorphic over R to the canonical lift of E_0.

Clearly

$$\mathbb{M}_{ord}\big(R[\![q]\!]\big) = \bigcup_{E_0/k \text{ ordinary}} \mathbb{M}\big(R[\![q]\!]\big)_{E_0}.$$

On the other hand, let us consider the map

$$\begin{aligned} \iota_{E_0} : qR[\![q]\!] \times \big(R[\![q]\!]^\times/\{\pm 1\}\big) &\to \mathbb{M}\big(R[\![q]\!]\big), \\ (u, v) &\mapsto \big(v^4 a_{4,E_0}(u), v^6 a_{6,E_0}(u)\big). \end{aligned} \tag{26}$$

Lemma 3.12. *Assume $j(E_0) \neq 0, 1728$ in k (that is to say, $A_0 B_0 \neq 0$). Then the map ι_{E_0} in* (26) *is injective, and its image is equal to $\mathbb{M}(R[\![q]\!])_{CL,E_0}$.*

Proof. We prove that $\operatorname{Im} \iota_{E_0} = \mathbb{M}(R[\![q]\!])_{CL,E_0}$. The inclusion of the first set into the second follows by Remark 3.11. Let us check the opposite inclusion. Assume $(a,b) \in \mathbb{M}(R[\![q]\!])_{CL,E_0}$, and set $A = a(0)$, $B = b(0)$. So $y^2 = x^3 + Ax + B$ is isomorphic over R to $y^2 = x^3 + a_{4,E_0}(0)x + a_{6,E_0}(0)$. Thus, $A = \lambda^4 a_{4,E_0}(0)$, $B = \lambda^6 a_{6,E_0}(0)$ for some $\lambda \in R^\times$, and without loosing generality, we assume that $\lambda = 1$. Let $j_{E_0} \in R[\![t]\!]$ be the Serre–Tate expansion of the j-invariant $j \in \mathbb{Z}_p[a_4, a_6, \Delta^{-1}]$. Since the j-invariant is unramified on the fine moduli spaces

$Y_1(N)$ $(N \geqslant 4)$ above points $\not\equiv 0, 1728 \bmod p$, it follows that $\frac{dj_{E_0}}{dt}(0) \in R^\times$. So the series $j_{E_0} - j_{E_0}(0) \in tR[\![t]\!]$ has a compositional inverse $\sigma(t) \in tR[\![t]\!]$. Notice that $j_{E_0}(0) = j(A, B)$. Let $u := \sigma(j(a,b) - j(A,B)) \in qR[\![q]\!]$. Then

$$j_{E_0}(u) - j_{E_0}(0) = j(a,b) - j(A,B),$$

hence $j_{E_0}(u) = j(a,b)$. Thus, the elliptic curves $y^2 = x^3 + ax + b$ and $y^2 = x^3 + a_{4,E_0}(u)x + a_{6,E_0}(u)$ have the same j-invariant, and therefore, there exists v in the algebraic closure of the fraction field of $R[\![q]\!]$ such that $a = v^4 a_{4,E_0}(u)$ and $b = v^6 a_{6,E_0}(u)$, respectively. From this point on, the rest of the argument is as in the proof of Lemma 3.5.

Injectivity of ι_{E_0} is proved along the same lines. $\square$

Let us set

$$\mathbb{M}\big(R[\![q]\!]\big)_{good} = \big\{(a,b) \in \mathbb{M}_{ord}\big(R[\![q]\!]\big):\ a, b \in R[\![q]\!]^\times\big\}.$$

Then we may consider the set of *good reduction solutions*

$$\mathcal{U}_{good} = \big\{(a,b) \in \mathbb{M}\big(R[\![q]\!]\big)_{good}:\ f(a,b) = 0\big\}.$$

Similarly, we may consider the set of solutions of f corresponding to E_0,

$$\mathcal{U}_{E_0} = \big\{(a,b) \in \mathbb{M}\big(R[\![q]\!]\big)_{E_0}:\ f(a,b) = 0\big\}.$$

Notice that, by Lemma 3.7, we have

$$\mathcal{U}_{E_0} \subset \mathbb{M}\big(R[\![q]\!]\big)_{CL,E_0}. \tag{27}$$

Clearly,

$$\mathcal{U}_{good} = \bigcup_{j(E_0) \neq 0, 1728, ss} \mathcal{U}_{E_0}, \tag{28}$$

where ss are the supersingular values. In order to study the set $\mathcal{U}_{good}$, it suffices to study the sets $\mathcal{U}_{E_0}$ for all E_0 with $j(E_0) \neq 0, 1728, ss$. This is the object of the next theorem.

Let us define, for $v \in R[\![q]\!]^\times$ and $\kappa \in \mathbb{Z}_+$, $\kappa \notin p\mathbb{Z}$, $z \in \mathbb{Z}_p$, $\alpha \in R$, the following pairs:

$$\begin{aligned} u_{\Gamma,E_0,v} &:= \big(v^4 a_{4,E_0}(0), v^6 a_{6,E_0}(0)\big) \in \mathbb{M}_{ord}\big(R[\![q]\!]\big), \\ u^z_{\Gamma,E_0,v,\kappa,\alpha} &:= \big(v^4 a_{4,E_0}\big(u^z_{m,\kappa,\alpha} - 1\big), v^6 a_{6,E_0}\big(u^z_{m,\kappa,\alpha} - 1\big)\big) \in \mathbb{M}_{ord}\big(R[\![q]\!]\big). \end{aligned}$$

Notice that, by Lemma 3.12, the maps

$$\begin{aligned} v &\mapsto u_{\Gamma,E_0,v}, \\ (\alpha, v) &\mapsto u^z_{\Gamma,E_0,v,\kappa,\alpha}, \end{aligned}$$

are injective. By Remark 3.8, $u_{\Gamma,E_0,v}$ does not depend on our choice of a basis of the Tate module $T_p(E_0)$. Also, multiplying a basis of the Tate module by an element $\lambda\in\mathbb{Z}_p^\times$ has the effect of replacing $u_{\Gamma,E_0,v,\kappa,\alpha}$ by $u_{\Gamma,E_0,v,\kappa,\lambda^2\alpha}$.

Theorem 3.13. *Consider the arithmetic heat equation given by*

$$f=f_q^1+\lambda f_{p,-2}^1+\epsilon f_{p,-2}^{1\phi},$$

where $\lambda\in R^\times$, $\epsilon\in pR$, and $z:=\frac{\epsilon}{\lambda}\in p\mathbb{Z}_p$. Let E_0 be an ordinary elliptic curve over k with $j\neq 0,1728$, $\kappa:=\frac{c_{E_0}}{\gamma_{E_0}}\cdot\lambda\in R^\times$, and $\mathcal{U}_{E_0}$ be the set of solutions corresponding to E_0. Then the following hold:

(1) *Assume $\kappa\notin\mathbb{Z}_+$. Then*

$$\mathcal{U}_{E_0}=\{u_{\Gamma,E_0,v}\mid v\in R[\![q]\!]^\times\}.$$

(2) *Assume $\kappa\in\mathbb{Z}_+$. Then*

$$\mathcal{U}_{E_0}=\{u^z_{\Gamma,E_0,v,\kappa,\alpha}\mid v\in R[\![q]\!]^\times,\ \alpha\in R\}.$$

Remark 3.14. According to the two cases above, we have corresponding canonical identifications $\mathcal{U}_{E_0}/R[\![q]\!]^\times\simeq 0$ and $\mathcal{U}_{E_0}/R[\![q]\!]^\times\simeq R$, where the latter is obtained via the α parameter. This and those identifications in (25) show, for instance, that there are canonical identifications between bad and good reduction solutions

$$\mathcal{U}_\zeta/R[\![q]\!]^\times\simeq\mathcal{U}_{E_0}/R[\![q]\!]^\times$$

of the arithmetic convection equations defined by

$$f_q^1+\xi\frac{\gamma}{c}f_{p,-2}^1\quad\text{and}\quad f_q^1+\xi\frac{\gamma_{E_0}}{c_{E_0}}f_{p,-2}^1,$$

respectively, where ξ is any element of $R^\times$.

Proof of Theorem 3.13. By Lemmas 3.9, 3.7, 3.12, and (27), $\mathcal{U}_{E_0}$ consists of all pairs $(v^4a_{4,E_0}(u),v^6a_{6,E_0}(u))$ with $v\in R[\![q]\!]^\times$, $u\in qR[\![q]\!]$, such that

$$\Psi_q(1+u)+\kappa\Psi_p(1+u)+\kappa z\phi\Psi_p(1+u)=0. \tag{29}$$

The conclusion follows from this exactly as in the case of bad reduction. $\square$

4. Instability

We recall the series in (20), and start with the following obvious remark: for any fixed $0 \neq \kappa \in \mathbb{Z}_+$, the function

$$\mathbb{Z}_p \times R \to R[\![q]\!],$$
$$(z, \alpha) \mapsto u^z_{a,\kappa,\alpha}$$

is continuous for the p-adic topologies of $\mathbb{Z}_p \times R$ and $R[\![q]\!]$. Morally speaking, this is a *stability* result for $\mathbb{G}_a$.

In deep contrast with this, we have the following *instability* result for $\mathbb{G}_m$ and $\mathbb{M}_\Gamma$:

Theorem 4.1. *Let $\kappa \in \mathbb{Z}_+$, $\kappa \notin p\mathbb{Z}$, $\alpha_0, \alpha \in R$, $\eta_0, \eta \in R^\times$, $z_0, z \in \mathbb{Z}_p$. Assume that one of the following holds:*

(1) $\alpha = \alpha_0$ *and* $z \neq z_0$,
(2) $\alpha \neq \alpha_0$ *and* $z = z_0$.

Then we have that

$$\eta u^z_{m,\kappa,\alpha} \not\equiv \eta_0 u^{z_0}_{m,\kappa,\alpha_0} \quad \text{mod } p \textit{ in } R[\![q]\!]. \tag{30}$$

Moreover, if $v_0, v \in R[\![q]\!]^\times$, then

$$u^z_{\Gamma,\eta,v,\kappa,\alpha} \not\equiv u^{z_0}_{\Gamma,\eta_0,v_0,\kappa,\alpha_0} \quad \text{mod } p \textit{ in } R[\![q]\!], \tag{31}$$

$$u^z_{\Gamma,E_0,v,\kappa,\alpha} \not\equiv u^{z_0}_{\Gamma,E_0,v_0,\kappa,\alpha_0} \quad \text{mod } p \textit{ in } R[\![q]\!]. \tag{32}$$

Under the hypothesis (1), (31) morally implies that for $z_0 = 0$, for instance, if α is a given "boundary condition," then no matter how close the "parameter" $z \neq 0$ is to 0, a solution $u^z_{\Gamma,\zeta\beta,v,\kappa,\alpha}$ of the arithmetic heat equation defined by $f^1_q + \lambda f^1_{p,-2} + \epsilon f^{1\phi}_{p,-2}$ $(z = \epsilon/\lambda)$ with "boundary condition" α is never close to a solution $u^0_{\Gamma,\zeta_0\beta_0,v_0,\kappa,\alpha}$ of the arithmetic convection equation $f^1_q + \lambda f^1_{p,-2}$ that has the same "boundary condition" $\alpha_0 = \alpha$. A similar interpretation holds for (32). Furthermore, under the hypothesis (2), (31) implies that given an arithmetic heat (or convection) equation, any two solutions corresponding to different "boundary conditions" $\alpha \neq \alpha_0$ are always far apart.

Proof. We assume hypothesis (1). The case when using (2) instead can be treated in a similar way.

We take $\frac{\partial}{\partial x}$ in the recurrence relation (19) for the $b_n(x, y)$s, and set $x = y = 0$. We then get

$$\frac{\partial b_n}{\partial x}(0,0) = \frac{\partial b_{n-1}}{\partial x}(0,0) \quad \text{for } n \geqslant 2.$$

Since $\frac{\partial b_1}{\partial x}(0,0) = -1$, we obtain that

$$\frac{\partial b_n}{\partial x}(0,0) = -1 \quad \text{for } n \geqslant 1.$$

In particular, viewing the b_ns as elements in $\mathbb{Z}[\![x,y]\!]$, for all $n \geqslant 1$, we may write

$$b_n(x,y) = b_n(0,y) - x + A_n(x,y)xy + B_n(x,y)x^2,$$

where $A_n, B_n \in \mathbb{Z}[\![x,y]\!]$. We deduce that

$$\begin{aligned} b_n(pz,p) - b_n(pz_0,p) = p(z-z_0) + p^2 z\big[A_n(pz,p) - A_n(pz_0,p)\big] + p^2(z-z_0)A_n(pz_0,p) \\ + p^2z^2\big[B_n(pz,p) - B_n(pz_0,p)\big] + p^2\big(z^2 - z_0^2\big)B_n(pz_0,p). \end{aligned}$$

Let us denote by $v_p : \mathbb{Q}_p^\times \to \mathbb{Z}$ the p-adic valuation. Since

$$v_p\big(A_n(pz,p) - A_n(pz_0,p)\big) \geqslant v_p(z-z_0) + 1,$$
$$v_p\big(B_n(pz,p) - B_n(pz_0,p)\big) \geqslant v_p(z-z_0) + 1,$$

we deduce that

$$v_p\big(b_n(pz,p) - b_n(pz_0,p)\big) = v_p(z) + 1 \quad \text{for all } n \geqslant 1. \tag{33}$$

Assume now that (30) is false, that is to say,

$$\eta u^z_{m,\kappa,\alpha} - \eta_0 u^{z_0}_{m,\kappa,\alpha} \in pR[\![q]\!].$$

We derive a contradiction. Setting $q = 0$ in this expression, we get that

$$\eta \equiv \eta_0 \mod p,$$

and it follows that

$$\frac{u^z_{m,\kappa,\alpha}}{u^{z_0}_{m,\kappa,\alpha}} \in 1 + pR[\![q]\!].$$

On the other hand,

$$\frac{u^z_{m,\kappa,\alpha}}{u^{z_0}_{m,\kappa,\alpha}} = \exp\left(\sum_{n\geqslant 0}\big(b_n(pz,p) - b_n(pz_0,p)\big)\alpha^{\phi^n}\frac{q^{\kappa p^n}}{\kappa p^n}\right).$$

Using the fact that the function $X \mapsto \log X$ maps $1 + pR[\![q]\!]$ into $pR[\![q]\!]$, we deduce that

$$\sum_{n\geqslant 0}\big(b_n(pz,p) - b_n(pz_0,p)\big)\alpha^{\phi^n}\frac{q^{\kappa p^n}}{\kappa p^n} \in pR[\![q]\!].$$

In particular, for any $n \geqslant 1$, we have that

$$v_p\left(\left(b_n(pz,p)-b_n(pz_0,p)\right)\frac{\alpha^{\phi^n}}{\kappa p^n}\right)\geqslant 1,$$

hence, by (33), that

$$v_p(z)+1+v_p(\alpha)-n\geqslant 1$$

for all $n\geqslant 1$. This is a contradiction.

We now prove (31) arguing again by contradiction. Let us assume that (31) is false, that is to say, that we have that

$$u^z_{\Gamma,\eta,v,\kappa,\alpha}\equiv u^{z_0}_{\Gamma,\eta_0,v_0,\kappa,\alpha}\quad \bmod p \text{ in } R[\![q]\!].$$

Let $\frac{1}{j_\infty}(q)\in\mathbb{Z}[\![q]\!]$ be the reciprocal of the Fourier expansion of the j-function. Then

$$\frac{1}{j_\infty}\left(\eta q u^z_{m,\kappa,\alpha}\right)\equiv\frac{1}{j_\infty}\left(\eta_0 q u^{z_0}_{m,\kappa,\alpha}\right)\quad \bmod p \text{ in } R[\![q]\!].$$

Now $\frac{1}{j_\infty}(q)$ has a compositional inverse in $qR[\![q]\!]$. We conclude that

$$\eta u^z_{m\kappa,\alpha}\equiv\eta_0 u^{z_0}_{m,\kappa,\alpha}\quad \bmod p,$$

which contradicts (30). Thus, the assumption is false, and (31) holds.

Another argument similar to the one above can be used to prove (32). □

Remark 4.2. Theorem 4.1 implies in particular that for any fixed $\kappa\in\mathbb{Z}_+$, $\kappa\notin p\mathbb{Z}$, $z\in\mathbb{Z}_p$, $v\in R[\![q]\!]$, $\alpha_0\in R$ the functions

$$\begin{aligned} &R\to R[\![q]\!]\times R[\![q]\!], && R\to R[\![q]\!]\times R[\![q]\!],\\ &\alpha\mapsto u^z_{\Gamma,\eta,v,\kappa,\alpha}, && \alpha\mapsto u^z_{\Gamma,E_0,v,\kappa,\alpha}, \end{aligned}\tag{34}$$

are discontinuous at α_0 for the p-adic topologies of R and $R[\![q]\!]\times R[\![q]\!]$. However, if

$$c_n: R[\![q]\!]\times R[\![q]\!]\to R\times R$$

($n\geqslant 1$) are the maps

$$c_n\left(\sum a_i q^i,\sum b_i q^i\right)=(a_n,b_n),$$

then the maps

$$\begin{aligned} &R\to R\times R, && R\to R\times R,\\ &\alpha\mapsto c_n\left(u^z_{\Gamma,\eta,v,\kappa,\alpha}\right), && \alpha\mapsto c_n\left(u^z_{\Gamma,E_0,v,\kappa,\alpha}\right), \end{aligned}\tag{35}$$

are continuous for the p-adic topologies. Indeed, the maps (34) are pseudo δ_p-polynomial in the sense of Definition 6.10 in [7]. In particular, we see that the family (indexed by $n \geqslant 1$) of maps (35) is *not* equicontinuous at any α_0.

5. Analogies between various equations

The following table summarizes some of the analogies between various arithmetic convection equations introduced in [7] and in the present paper. We freely employ the notation of [7], and that introduced here.

Variety	Equation	Stationary solutions	Solutions with fixed behavior at infinity
$\mathbb{G}_a$	$\delta_q + \lambda\phi_p - \lambda$	$\mathcal{U}_0 = \mathbb{Z}_p$	$\mathcal{U}_1 \simeq R$ or 0
$\mathbb{G}_m$	$\psi_q + \lambda\psi_p$	$\mathcal{U}_0 = \mu(R)$	$\mathcal{U}_1 \simeq R$ or 0
$E = \mathrm{Tate}(\beta q)$	ψ^1_{pq}	$\mathcal{U}^1_0$	$\mathcal{U}^1_1 \simeq R$ or 0
E/R, a CL curve	$\psi^1_q + \lambda\psi^1_p$	$\mathcal{U}_0 = \cap p^n E(R)$	$\mathcal{U}_1 \simeq R$ or 0
$\mathbb{M}_\Gamma$	$f^1_q + \lambda f^1_{p,-2}$	$\mathcal{U}_0 = \emptyset$ (bad reduction) $\mathcal{U}_0 = \mathbb{M}(R)_{CL}$ (good reduction)	$\mathcal{U}_\zeta / R[\![q]\!]^\times \simeq R$ or 0 (bad reduction) $\mathcal{U}_{E_0} / R[\![q]\!]^\times \simeq R$ or 0 (good reduction)

In the table above, the set of stationary solutions $\mathcal{U}_0$ for $\mathbb{M}_\Gamma$, in the bad reduction case, is defined as the solutions in $\mathcal{U}_{bad} \cap \mathbb{M}_{ord}(R)$; the latter set is, of course empty. Also, the set $\mathcal{U}^1_0$ of *stationary solutions* for $E = \mathrm{Tate}(\beta q)$, defined in [7] consists of all solutions in the Manin kernel; cf. [7] for details. The alternative "R or 0" in the last column depends on the value of λ (for all cases except $E = \mathrm{Tate}(\beta q)$) respectively β (in the case of $E = \mathrm{Tate}(\beta q)$).

A similar table can be provided for arithmetic heat equations of order 2 (in which E/R is, this time, non-CL). On the other hand, as was shown in this paper, the analogy breaks down for wave equations: these wave equations exist (and have interesting solutions) for $\mathbb{G}_a$, $\mathbb{G}_m$, and E, but do not exist for $\mathbb{M}_\Gamma$.

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